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Transaction Costs in an Overlapping Generations Model

Mohanad Ismael[1]

ABSTRACT

We study the stability properties of a Diamond (1965) overlapping generations model in which agents have to pay transaction costs related to the capital accumulated. In particular, these costs depend positively on the amount of individual.s savings. At .rst, we show that under standard conditions, the feasible path may be dynamically inefficient (efficient) if there is an over-accumulation (under-accumulation) of capital with respect to Golden Rule. Namely, the introduction of transaction costs reduces the Golden Rule level of saving comparing to the standard model. It is also shown that the stationary equilibrium is determinate. Further, transaction costs promote the emergence of cycles of period two and therefore acts as a destabilizing factor. The analytical findings are completed by a numerical example.

1 INTRODUCTION

It is well known that transaction costs in asset and stock markets are considered as important factors in investment decision. The presence of transaction costs leads to inefficient portfolio diversifcation and distorts buyers' positions, and without these costs the portfolio choice would be a segment of the existing assets.

Transaction costs might consist of, among others, communication costs, time costs, government fees, stamp taxes, information and search costs, administration costs and brokerage commissions. Throughout the literature, authors are usually interested in studying the in.uence of transaction costs on portfolio choice and stock pricing. In addition, they also seek to explain why only a sub-set of households takes positions in the stock market. They perform portfolio models based on stock market transaction costs in order to justify the observed household's participation rate in the data.[2]

The study of transaction costs seems to be plausible and relevant. However, the effect of transaction costs on economic stability and capital accumulation has not been widely treated in literature yet. This paper attempts to fill this gap by introducing transaction costs related to savings in a standard Diamond overlapping generations (thereafter OLG) model. In contrast to infinite-horizon model, the use of OLG framework allows formulating the saving function ex-plicitly. Transaction costs would affect capital accumulations through savings and therefore in.uence the allocation of resources across the generations. In addition, it allows decision makers to submit policies which might be specific to each life generation. This justifies introducing the costs into OLG model instead of Ramsey infinite-horizon framework.

In the model of Diamond, agents live two periods: youth and adulthood. In the first period, young agents supply labor inelastically and allocate their wage income between consumption and savings. When they are old, they are retired and consume their savings entirely.[3] In this paper, it is assumed that young agents have to pay transaction costs related to their level of capital accumulation and these costs are increasing.

The main objective is to analyze the impact of these costs on dynamic efficiency and on economic stability and cycles. In addition, the robustness of our results is shown numerically by considering both separable and non-separable utility function.

Considering cycles is important because it represents a mirror of the economy. It helps people to understand what is occuring in the economy and suitable policies have to be applied by the decision makers. Further, it is pleasing to determine whether the qualitative performance of the dynamic system continues under different exogenous forces.

Our first result states that, under standard conditions, the steady state may be dynamically inefficient (efficient) in terms of consumption if there is an over-accumulation (under-accumulation) of capital with respect to the modified Golden Rule. In other words, the net returns of capital are lower (higher) than the gross rate of population growth.

Contrary to standard OLG model, the steady state can change its stability through cycles of period two. The emergence of these cycles requires a su¢ ciently high sensitivity of transaction costs with respect to savings, a low elasticity of marginal utility with respect to future consumption, high elasticity of marginal utility with respect to current consumption, and a high first-period consumption share. This result is comparable to Galor and Ryder (1989) but for different circumstances.

The intuition of these cycles is given as follows: assume that the level of current capital increases from its steady state value. This leads wage income to rise, which induces more capital accumulation. However, there are some factors that influence capital accumulation negatively such as the presence of high transaction costs associated with savings, the existence of low elasticity of marginal utility of future consumption, high sensitivity of marginal utility of current consumption, and high consumption share enforce agents to accumulate low capital. Cycles of period two are obtained whenever the latter e¤ects dominate the former one. Therefore, higher transaction costs makes the appearance of cycles of period two more likely.

The remainder of the paper is organized as follows. In section two, we present the optimization problem of households and firms. The intertemporal equilibrium is presented in section three. We present the steady state analysis in section four. The dynamic efficiency of the intertemporal equilibrium is studied in section five, while in section six, we present the local dynamics. We discuss the results in section seven. A numerical example is located in section eight. Sections nine and ten are the conclusion and the appendix respectively.

2. The model

Consider a non-monetary overlapping generations economy with identical agents who live two periods. In each period t, N_t individuals are born and they live for two periods 'young and old'. In this model, at each period, there is a unique good that can be either consumed or invested. In the first period, agents are endowed with one unit of labor which is supplied inelastically to .rms. They choose their amounts of consumption and saving along with income. In addition, agents have to pay variable costs 'transaction costs' related to saving amount. One example, the utilization of investment advisor will charge you advisor fees based on the value of your portfolio. In addition, broker-dealer can charge you some additional fees such making purchasing securities, opening an account, account transfer, etc. Further, addition fees required to purchase corporate bonds is from 0.05 per cent to 3 per cent of the amount invested.[4] In the second period, they do not work and their income comes from the return of first-period saving.

Given the real wage wt and the real return factor R_{t+1}, agents allocate savings and consumptions for both periods to maximize the following intertemporal preferences:

$$U(c_t, d_{t+1}) \tag{1}$$

Subject to the constraints

$$c_t + s_t + \xi(s_t) \le w_t \tag{2}$$

$$d_{t+1} \le R_{t+1} s_t \tag{3}$$

$$c_t \ge 0, d_{t+1} \ge 0 \text{ for all } t \ge 0$$

where c_t, d_{t+1} is the consumption in the first 'young' and second 'old' period respectively, s_t is the saving and $\xi(s_t)$ is the transaction cost associated with saving.[5]

Assumption 1 $U(c_t, d_{t+1})$ is strictly increasing with respect to each argument $U_1(c_t, d_{t+1}) > 0, U_2(c_t, d_{t+1}) > 0$ with $U_{11}(c_t, d_{t+1}) < 0, U_{22}(c_t, d_{t+1}) < 0, C^2$ over the interior of the set $R_+^2 = [0, +\infty)x[0, +\infty)$. Additionally, $\lim_{d_{t+1} \to +\infty} U_1(c_t, d_{t+1}) / U_2(c_t, d_{t+1}) = +\infty$ and $\lim_{d_{t+1} \to 0} U_1(c_t, d_{t+1}) / U_2(c_t, d_{t+1}) = 0$ for all $c_t, d_{t+1} > 0$. Furthermore, $\lim_{c_1 \to 0^+} U_1(c_t, d_{t+1}) / U_2(c_t, d_{t+1}) = +\infty$ and $\lim_{c_1 \to +\infty} U_1(c_t, d_{t+1}) / U_2(c_t, d_{t+1}) = 0$ for all $c_t, d_{t+1} > 0$.[6]

For future reference, we propose some necessary elasticities: the elasticity of marginal utility with respect to the first and second arguments are, respectively, $\varepsilon_{11} \equiv u_{11}(c,d)c/u_1(c,d) < 0$, $\varepsilon_{22} \equiv u_{22}(c,d)d/u_2(c,d) < 0$. The cross elasticities in consumption are $\varepsilon_{21} \equiv u_{21}(c,d)c/u_2(c,d)$, $\varepsilon_{12} \equiv u_{12}(c,d)d/u_2(c,d)$.

Assumption 2 The cost function is increasing in its argument $\xi'(s_t) > 0$ and concave $\xi''(s_t) < 0$.

In the real world, the amount of transaction costs gets higher if investment increases simply because brokerage fees are calculated as a (flat rate) percentage of the amount invested. The concavity of transaction costs stems from the fact that the time (or administrative fees) required to purchase two assets is not twice the time (or fees) needed to purchase one asset.[7] In order to simplify the notation, we assume[8] $\varphi(s_t) = s_t + \xi(s_t)$.

The Lagrangian function for the household problem is:

$$L = u(c_t, d_{t+1}) + \lambda_t (w_t - c_t - \varphi(s_t)) + \mu_t (R_{t+1} s_t - d_{t+1}) \tag{4}$$

The first-order conditions with respect to c_t , d_{t+1} and s_t are, respectively:

$$u_1(c_t, d_{t+1}) = \lambda_t \tag{5}$$

$$u_2(c_t, d_{t+1}) = \mu_t \tag{6}$$

$$\mu_t R_{t+1} = \varphi'(s_t)\lambda_t \tag{7}$$

This gives

$$u_1(c_t, d_{t+1})/u_2(c_t, d_{t+1}) = R_{t+1}/\varphi'(s_t) \tag{8}$$

The LHS is simply the marginal rate of substitution between consumption 'today' and consumption 'tomorrow'. Given the existence of increasing cost, the associated marginal rate of substitution is smaller than that of the standard Diamond model. In other words, the interest rate factor is higher than that of the standard Diamond without costs, which implies that agents accumulate less capital. The model of Diamond (1965) is obtained by setting $\varphi(s_t) = s_t$.

On the production side, a representative firm uses labour and capital to produce final goods using constant returns-to-scale technology $AF(K_t, L_t)$ with $A>0$ a productivity scaling factor. Let $a_t = K_t / L_t$ be the capital stock per labour unit, then the production function can be written as $Af(a_t) = AF(a_t, 1)$.

Assumption 3 Let $a \geq 0$, the technology $f(a_t)$ is continuous and differentiable. It is increasing $f'(a) > 0$ and concave $f''(a) < 0$. Furthermore,

$f(0)=0$, $\lim_{a\to 0^+} f'(a)=+\infty$ and $\lim_{a\to +\infty} f'(a)=0$

Each representative firm takes real wages w_t and rental prices R_t as given. If we set $\rho(a_t)=f'(a_t)$ and $\omega(a_t)=f(a_t)-a_t\, f'(a_t)$ then the competitive equilibrium conditions for profit maximisation require that the real interest rate and the real wage satisfy:

$$R_t = A\rho(a_t) \text{ and } w_t = A\omega(a_t) \tag{9}$$

Thus, we can deduce that the elasticity of interest rate $a\rho'(a)=\rho(a)= -(1-a)/\sigma<0$ and the elasticity of wage $a\omega'(a)/\omega(a) = \alpha/\sigma>0$, with $\sigma\in(0,+\infty)$ is the elasticity of capital-labour substitution, while $\alpha\in(0,1)$ is the capital share in total income.

3. Intertemporal equilibrium

The number of households in each generation grows at a constant rate $n>-1$ such that $1+n = N_{(t+1)}/N_t$, where N_t is the number of people born at time t. In equilibrium, three markets clear:

The capital market clears according to capital-accumulation equation: $K_{(t+1)}=N_t\, s_t$

The labour market clears: $L_t=N_t$

By Walras' law, the output market also clears: $N_t\,(c_t+\varphi(s_t))+N_{(t-1)}d_t=AF(K_t,L_t)$

From market clearing conditions, one can demonstrate that:

$$s_t = (1+n)\,a_{t+1} \tag{10}$$

Substituting (10) and condition (9) together with the binding budget constraints (2) and (3) into (8) yields the following one-dimensional dynamic system of a.

$$\frac{u_1\left[A\omega\left(a_t\right)-\varphi\left[a_{t+1}\left(1+n\right)\right],A\rho\left(a_{t+1}\right)a_{t+1}\left(1+n\right)\right]}{u_2\left[A\omega\left(a_t\right)-\varphi\left[a_{t+1}\left(1+n\right)\right],A\rho\left(a_{t+1}\right)a_{t+1}\left(1+n\right)\right]A}-\frac{\rho\left(a_{t+1}\right)}{\varphi'\left[a_{t+1}\left(1+n\right)\right]}=0 \tag{11}$$

At the steady state $a_{(t+1)}=a_t=a$ so the dynamic system (11) becomes:

$$\frac{u_1\left[A\omega\left(a\right)-\varphi\left[a\left(1+n\right)\right],A\rho\left(a\right)a\left(1+n\right)\right]}{u_2\left[A\omega\left(a\right)-\varphi\left[a\left(1+n\right)\right],A\rho\left(a\right)a\left(1+n\right)\right]A}-\frac{\rho\left(a\right)}{\varphi'\left[a\left(1+n\right)\right]}=0 \tag{12}$$

To simplify the analysis, we follow the method initiated by Cazzavillan *et al* (1998), by using a scaling parameter A in order to give conditions for the existence of a normalised steady state a = 1.

Proposition 1 *Under assumptions 1 - 3, a = 1 is a steady state of the dynamic system (11) if and only if there exists a scaling parameter A such that* $A > \hat{A} \equiv \varphi[1+n]/\omega(1)$ *and satisfies:*

$$\frac{u_1\left[A\omega(1)-\varphi[1+n], A\rho(1)(1+n)\right]}{u_2\left[A\omega(1)-\varphi[1+n], A\rho(1)(1+n)\right]A} = \frac{\rho(1)}{\varphi'[1+n} \tag{13}$$

The scaling parameter A is a unique solution of (13) if and only if:

$$(\varepsilon_{11}-\varepsilon_{21})/\gamma+\varepsilon_{12}-\varepsilon_{22}-1<0 \text{ for all } A^9.$$

Proof The solution a = 1 is a steady state if and only if (13) is verified. Moreover, the positivity of first-period consumption requires $A > \hat{A} \equiv \varphi[1+n]/\omega(1)$, so $A \in (\hat{A}, +\infty)$. Let us call the LHS:

$$G(A) = \frac{u_1\left[A\omega(1)-\varphi[1+n], A\rho(1)(1+n)\right]}{u_2\left[A\omega(1)-\varphi[1+n], A\rho(1)(1+n)\right]A}$$

Since it is a continuous function, then based on Assumption 1 it is easy to show that $\lim_{A\to\hat{A}} G(A) = +\infty$ and $\lim_{A\to+\infty} G(A) = 0$. Since the RHS is a positive constant, there thus exists a steady state at a = 1. Concerning the uniqueness of A, it is enough to show that $G(A)$ is monotonic (decreasing), i.e., $(\varepsilon_{11}-\varepsilon_{21})/\gamma+\varepsilon_{12}-\varepsilon_{22}-1<0$ is satisfied for all A.

Assumption 4 The utility function $u(c_t, d_{(t+1)})$ is homogeneous of degree one.

Corollary 1 Under Assumptions 1 - 4, a = 1 is a steady state for the dynamic system (11) if and only if there exists a scaling parameter A such that $A > \hat{A} \equiv \varphi[1+n]/\omega(1)$ and satisfies:

$$\frac{u_1\left[\frac{A\omega(1)-\varphi[1+n]}{A\rho(1)(1+n)}, 1\right]}{u_2\left[\frac{A\omega(1)-\varphi[1+n]}{A\rho(1)(1+n)}, 1\right]A} = \frac{\rho(1)}{\varphi'[1+n]} \tag{14}$$

In this case, the scaling parameter A is unique.

Proof Since the utility function is homogenous of degree one, equality (12) can be written as (14). The solution a = 1 is a steady state if and only if equality (14) is satisfied. Notice that the RHS does not change and $A \in (\hat{A}, +\infty)$. If we denote the LHS by Q:

$$Q(A) \equiv \frac{u_1\left[\frac{\omega(1)}{\rho(1)(1+n)} - \frac{\varphi[1+n]}{A\rho(1)(1+n)}, 1\right]}{u_2\left[\frac{\omega(1)}{\rho(1)(1+n)} - \frac{\varphi[1+n]}{A\rho(1)(1+n)}, 1\right] A} \tag{15}$$

Using Assumption 1, it is easy to show that $\lim_{A\to\hat{A}} G(A) = +\infty$ and $\lim_{A\to+\infty} G(A) = 0$. Further, from the homogeneity property of the utility function, one can prove that $u_{21} > 0$. Consequently, a direct inspection of (15) gives that $Q'(A) = \left[[\varphi[1+n](u_2u_{11} - u_1u_{21})/A\rho(1)(1+n)] - u_1u_2\right]/(u_2A)^2 < 0$ which implies that there is a unique scaling parameter A satisfying (14).

Assumption 5 The utility function is separable.

Corollary 2 Let Assumptions 1 - 3 and 5 be satisfied, then $a = 1$ is a steady state for the dynamic system (11) if there exists a scaling parameter A such that $A > \hat{A} \equiv \varphi[1+n]/\omega(1)$ and satisfies:

$$\frac{v'[A\omega(1) - \varphi[1+n]]}{v'[A\rho(1)(1+n)]}\frac{1}{A} = \frac{\beta\rho(1)}{\varphi'[1+n]} \tag{16}$$

Proof In a separable case, the household's problem is simplified at the steady state to (16). As before, let us call the LHS:

$$\Pi(A) \equiv \frac{v'[A\omega(1) - \varphi[1+n]]}{v'[A\rho(1)(1+n)]}\frac{1}{A}$$

and A belongs to $(\hat{A}, +\infty)$, then, based on Assumption 1, we have $\lim_{A\to\hat{A}} G(A) = +\infty$ and $\lim_{A\to+\infty} G(A) = 0$. In order to show the existence of a unique A, it is easy to demonstrate that $\Pi(A)$ is always decreasing i.e. $\varepsilon(1/\gamma - 1) - 1 < 0$, where ε is the elasticity of the marginal utility of consumption. Throughout the paper, it is assumed that the above propositions hold for each configuration.

4. Dynamic efficiency

In this section, we analyse the dynamic efficiency of the steady state. Before passing through the efficiency analysis, let us define the following useful elasticities: the elasticity of transaction costs with respect to savings, the elasticity of marginal transaction costs $\eta_1 = \varphi'(s)s/(\varphi(s)) > 0$, the elasticity of marginal transaction costs, $\eta_2 = \varphi''(s)s/\varphi'(s) < 0$.

Using the intertemporal equilibrium conditions bovea, we obtain the following stationary resource constraint:

$$c + \frac{d}{1+n} = \sum(a) \tag{17}$$

with

$$\sum(a) \equiv Af(a) - \varphi\left[(1+n)a\right] \tag{18}$$

the net production and the LHS simply the stationary aggregate consumption.[10]

Assumption 6 Assume that $-(1-\alpha)/\sigma < \eta_2$.

This assumption is necessary to confirm that net production $\Sigma(a)$ is concave. Subsequently, this ensures the existence of a unique positive capital-labour ratio that maximises the net production and so allocates the maximum level of consumption.[11] In order to characterise the modified Golden Rule capital-labour ratio, we need to make the following concavity assumption, where simply the concavity of the production function $f(a)$ is higher than that of the cost function $\varphi[(1+n)a]$. Otherwise, investment would not occur. Technically, the importance of Assumption 7 is that it allows for equality (19) to have a unique solution.

Assumption 7 Assume that:

$$\begin{array}{lcl} \lim\limits_{a\to 0^+} Af'(a) & > & \lim\limits_{a\to 0^+} (1+n)\,\varphi'\left[(1+n)\,a\right] \\ \lim\limits_{a\to +\infty} Af'(a) & < & \lim\limits_{a\to +\infty} (1+n)\,\varphi'\left[(1+n)\,a\right] \end{array}$$

Following Phelps (1965) and Diamond (1965), we define the modified Golden Rule level of the capital-labour ratio.[12]

Definition 1 (Modified Golden Rule) Under Assumptions 6 and 7, there exists a unique positive capital stock per young agents such that:

$$\frac{Af(\bar{a})}{\varphi'\left[(1+n)\bar{a}\right]} \tag{19}$$

with $\bar{a}$ the modified Golden Rule capital-labour ratio.

The modified Golden Rule (19) determines the level of capital in which the net marginal productivity of capital equals the gross rate of population growth. Modified Golden Rule capital does not depend on consumption allocations in both periods. At the same time, this level of capital provides the highest level of consumption. On the other side, the equilibrium level of capital which satisfies the modified Golden Rule is lower than that in the standard model without costs.

Proposition 2 *Under Assumptions 6 - 7, there is a unique optimal stationary path, the modified Golden Rule, which is characterised by* $a = \overline{a}$ *and by* $\overline{c}, \overline{d}$ *satisfying the following conditions:*

$$\bar{c} + \frac{\bar{d}}{1+n} = \Sigma(\bar{a}) \tag{20}$$

$$u_1(\bar{c}, \bar{d}) = (1+n)\, u_2(\bar{c}, \bar{d}) \tag{21}$$

Proof. The maximum of $\Sigma\ \overline{a}$ [13] is satisfied using the modified Golden Rule (19) and the optimal allocation of first-period and second-period consumption that maximises the household's preferences (1) under the constraint (20), is illustrated by the first-order necessary condition (21).

Definition 2 (Feasible path of capital) A sequence of capital stock per young agents $a_t \geq 0$ is a feasible path if the corresponding production net of investment i.e., $\sum(a_t, a_{t+1}) \equiv Af(a_t) - \varphi\left[(1+n)a_{t+1}\right] \geq 0$ is non-negative for all $t > 0$.

Definition 3 (Efficiency) A feasible sequence of capital per young agents $\{a_t\}_{t\geq 0}$ is efficient if it is impossible to raise an agent's consumption at one date without reducing it at another date, i.e., if there does not exist another feasible path $\{\ddot{a}_t\}_{t\geq 0}$ with $\ddot{a}_0 = a_0$ such that:

$$(i)\ \Sigma(\ddot{a}_t, \ddot{a}_{t+1}) \geq \Sigma(a_t, a_{t+1}) \text{ for all } t \geq 0.$$
$$(ii)\ \Sigma(\ddot{a}_t, \ddot{a}_{t+1}) > \Sigma(a_t, a_{t+1}) \text{ for some } t \geq 0.$$

Now, let us consider a feasible path of the capital-labour ratio where this path converges to the normalised steady state value $a^* = 1$. Then, we deduce the following result:

Proposition 3 *Under Assumptions 6 and 7, a feasible path which converges to a limit* $a^* > 0$ *is inefficient when* a^* *satisfies an over-accumulation of capital, while it is efficient whenever* a^* *satisfies an under-accumulation of capital.*

Proof. See Appendix (B).

Dynamic efficiency is the same as Pareto optimality in terms of aggregate consumption. Proposition 3 states that there is an under-accumulation (or an over-accumulation) of capital compared to the modified Golden Rule level, if the net capital rate of return is higher (or lower) than the gross rate of population growth.

In other words, in the under-accumulation capital case (case i), it is impossible to increase consumption in one period t without reducing it in any other period, while in the over-accumulation capital case (case ii) agents can increase their consumption without reducing it in another period.

Corollary 3 Let Assumptions 1-3, 6 and 7 be satisfied, then compared to the standard Diamond model the steady state is characterised by an under-accumulation of capital.

Proof. Let us define the Golden-Rule level of capital in Diamond (1965) as a^D, where a^D satisfies $Af(a^D)=1+n$. However, the modified Golden-Rule level of capital in the presence of costs is such as $Af'(\overline{a})=(1+n)\varphi'\left[(1+n)\overline{a}\right]$. Given that $\varphi'\left[(1+n)\overline{a},\right]>1$ this implies that $Af'(\overline{a})=(1+n)\varphi'\left[(1+n)\overline{a}\right]$. Therefore, capital accumulation with costs $\overline{a}$ is lower than that in the standard Diamond model without costs $\overline{a}^D$.

5. Local dynamics

In this section, we study the economic stability locally around the normalised steady state a=1. It is demonstrated that the introduction of transaction costs in a standard OLG à la Diamond affects the appearance of cycles of period two. Linearising the dynamic equation (11) around the steady state a=1 yields the following eigenvalue $J \equiv da_{t+1}/da_t$:

$$J=\frac{\alpha\left(\frac{\varepsilon_{12}}{\eta_1}\frac{1}{1-\gamma}-\frac{\varepsilon_{11}}{\gamma}\right)}{\sigma\left(2\varepsilon_{12}-\varepsilon_{22}-\eta_1\frac{1-\gamma}{\gamma}\varepsilon_{11}+\eta_2\right)+(1-\alpha)\left(1-\varepsilon_{12}+\varepsilon_{22}\right)} \tag{22}$$

Notice that a_t is a predetermined variable, therefore the steady state of system (11) is determinate. Further, the steady state is stable whenever the unique eigenvalue belongs to the interior of the unit circle, i.e. belongs to the interval (-1, 1). The second-order conditions (SOCs) associated with the household problem imply that:[14]

$$2\varepsilon_{12}-\varepsilon_{22}-\eta_1\frac{1-\gamma}{\gamma}\varepsilon_{11}+\eta_2>0$$

A sufficient condition for the emergence of cycles of period two is generically $J(\sigma)=-1$. This holds at $\sigma=\sigma^F$, where

$$\sigma^F\equiv-\frac{\alpha\left(\frac{\varepsilon_{12}}{\eta_1}\frac{1}{1-\gamma}-\frac{\varepsilon_{11}}{\gamma}\right)+(1-\alpha)\left(1-\varepsilon_{12}+\varepsilon_{22}\right)}{2\varepsilon_{12}-\varepsilon_{22}-\eta_1\frac{1-\gamma}{\gamma}\varepsilon_{11}+\eta_2} \tag{23}$$

Before going on, we present some critical values for η_1, γ, ε_{22} and ε_{11}.

$$\begin{pmatrix} \eta_1^* \\ \gamma^* \\ \varepsilon_{11}^* \\ \varepsilon_{22}^* \end{pmatrix} = \begin{pmatrix} \varepsilon_{12}/\left[(1-\gamma)\left(\frac{\varepsilon_{11}}{\gamma} - \frac{1-\alpha}{\alpha}(1-\varepsilon_{12}+\varepsilon_{22})\right)\right] \\ \alpha\varepsilon_{11}/\left[(1-\alpha)(1-\varepsilon_{12}+\varepsilon_{22})\right] \\ (1-\alpha)(1-\varepsilon_{12}+\varepsilon_{22})/\alpha \\ \varepsilon_{12}-1 \end{pmatrix} \tag{24}$$

In the next proposition, we present the sufficient conditions for the appearance of cycles of period two.

Proposition 4 *In view of Assumptions 1 - 3 together with (24), flip bifurcation exists when σ is close to σ^F if one of the following conditions holds:*

1. For $\varepsilon_{11} > \varepsilon_{11}^*$, $\varepsilon_{22} < \varepsilon_{22}^*$ and $\gamma > \gamma^*$ with either: (i) $\varepsilon_{12} > 0$, $\eta_1 > \eta_1^*$; or (ii) $\varepsilon_{12} < 0$, $\eta_1 > 0$.

2. For $\varepsilon_{12} < 0$, $\varepsilon_{22} < \varepsilon_{22}^*$ and $\eta_1 < \eta_1^*$ with either: (i) $\varepsilon_{11} < \varepsilon_{11}^*$, for all $\gamma > 0$, or (ii) $\varepsilon_{11} > \varepsilon_{11}^*$, $\gamma < \gamma^*$.

3. For $\varepsilon_{12} < 0$, $\varepsilon_{22} > \varepsilon_{22}^*$ and $\eta_1 < \eta_1^*$ for all $\gamma > 0$ and $\varepsilon_{11} < 0$.

Proof. See Appendix C.

This proposition shows that flip cycles arise for different configurations concerning agents' preferences, the cost function and the first-period consumption share. In order to study the mechanism behind the emergence of these cycles, let us initially focus on the Benchmark model without costs.

6. Discussion of the result

In order to understand the role of transaction costs on economic stability, let us investigate the cases without costs with non-separable and separable preferences, respectively. The key difference between these types of preferences is that in the separable case, the amount of good x consumed by agents does not have any influence on the utility gained from consuming any amount of good *y*, while it does affect the non-separable case. Each applied study relating to consumption presumes that preferences are separable. For instance, data on car purchases are used independently of other consumption choices; and data on consumption good *x* in one year are used without considering intertemporal consumption choices. Such analysis is based on a separable preference. We argue that it is impossible for an empirical paper to avoid the separability assumption. At the same time, no one can deny that the separable preference case is a special case of the non-separable one.

6.1 Benchmark model

6.1.1 Non-separable utility

We recover the basic model studied by Diamond (1965), where agents do not pay transaction costs, by setting $\eta_1=1$ and $\eta_2=0$. Then, $J(\sigma)=-1$ holds at $\sigma=\sigma^F_{BM}$ where:

$$\sigma^F_{BM} \equiv -\frac{\alpha\left(\varepsilon_{12}\frac{1}{1-\gamma}-\frac{\varepsilon_{11}}{\gamma}\right)+(1-\alpha)(1-\varepsilon_{12}+\varepsilon_{22})}{2\varepsilon_{12}-\varepsilon_{22}-\frac{1-\gamma}{\gamma}\varepsilon_{11}} \tag{25}$$

where $2\varepsilon_{12}-\varepsilon_{22}-\frac{1-\gamma}{\gamma}\varepsilon_{11}>0$ by SOCs. Given the critical values $\varepsilon^{bb}_{22} \equiv \varepsilon_{12}-1$, $\varepsilon^b_{12} \equiv -(1-\alpha)/(2\alpha-1)$, $\varepsilon^b_{22} \equiv -\varepsilon_{12}(2\alpha-1)/(1-\alpha)-1$, $\varepsilon^b_{11} \equiv [\varepsilon_{12}\gamma/(1-\gamma)]+[(1-\alpha)(1-\varepsilon_{12}+\varepsilon_{22})\gamma/\alpha]$ and $\gamma^b \equiv [\alpha\varepsilon_{12}/(1-\alpha)(1-\varepsilon_{12}+\varepsilon_{22})]+1$, then the positivity of σ^F_{BM} in (25) requires one of the following conditions:

1. $\varepsilon_{11}>\varepsilon^b_{11}$ and $\gamma<\gamma^b$ for either (i) $\varepsilon_{12}>\varepsilon^b_{12}$, $\varepsilon_{22}<\varepsilon^{bb}_{22}$, or (ii) $0<\varepsilon_{12}<\varepsilon^b_{12}$ and $\varepsilon_{22}<\varepsilon^b_{22}$.
2. $\varepsilon_{11}>\varepsilon^b_{11}$ and $\varepsilon_{12}<0$ for either (i) $\varepsilon_{22}<\varepsilon^{bb}_{22}$ for all $\gamma>0$ or (ii) $\varepsilon_{22}>\varepsilon^{bb}_{22}$ and $\gamma>\gamma^b$.

Cycles of period two appear if capital increases in the current period and then decreases in the following period. One can easily observe from (25) that the appearance of cycles of period two depends on agents' preferences and on first-period consumption shares. Notice that a low (resp. high) ε_{22} means that as second-period consumption increases, its marginal utility declines significantly (resp. slightly). In addition, a high (resp. low) ε_{11} implies that as first-period consumption increases, its marginal utility declines slightly (resp. heavily).

The intuition for the existence of cycles of period two is the following: Focus on case 1 and assume that K_t increases from its steady state value. Then, w_t augments, which induces greater capital accumulation $K_{(t+1)}$. The presence of a sufficiently high ε_{11} and a small ε_{22} encourages agents to raise current consumption and to reduce future consumption, and thus to accumulate less capital. However, the presence of a small γ leads agents to consume less today and to accumulate more capital. Therefore, cycles of period two require that the former effect (ε_{11} and ε_{22}) dominates the effects of γ and w_t. As a result, an increase in K_t is followed by a decline in capital accumulation $K_{(t+1)}$.

In case 2, the economic intuition is essentially the same.

6.1.2 Separable utility

The separability of preferences can be obtained by setting $\varepsilon_{12}=0$. For simplicity, it is supposed that agents have the same utility in both periods, then flip cycles emerge at $\sigma^F_{BM,S}$ where:

$$\sigma^F_{BM,S} \equiv \frac{\alpha\varepsilon/\gamma - (1-\alpha)(1+\varepsilon)}{-\varepsilon\left(1+\frac{1-\gamma}{\gamma}\right)} \tag{26}$$

where ε is the elasticity of marginal utility in consumption. Thus the elasticity of intertemporal substitution in consumption is given by $-1/\varepsilon$. Note that if we assume high substitutability between consumption in both periods, that is, $1 + \varepsilon > 0$, then the result of Diamond (1965) is obtained with $\sigma^F_{BM,S} < 0$, which implies a stable steady state. However, since it is not the case here, then $\sigma^F_{BM,S} > 0$ for $\varepsilon < \min(-1, \varepsilon^{sb})$ and $\gamma > \gamma^{sb}$ with $\varepsilon^{sb} \equiv (1-\alpha)/(2\alpha-1)$ and $\gamma^{sb} \equiv \alpha\varepsilon/((1-\alpha)(1+\varepsilon))$. Hence, the income effect dominates the substitution effect and thus agents are not interested in future consumption. The presence of a high consumption share, together with a low elasticity of substitution, allows agents to accumulate less capital. This means that a rise in K_t in the current period would be followed by a decline in $K_{(t+1)}$ in the next period.

Furthermore, the eigenvalue (22) is simplified to:[15]

$$J = \frac{\alpha}{\sigma} \frac{-\varepsilon}{\frac{\gamma(1-\alpha)(1+\varepsilon)}{\sigma} - \varepsilon} \tag{27}$$

Equation (27) argues that the unique equilibrium path of the standard Diamond (1965) model is recovered as $\sigma > \alpha$.[16] However, Nourry (2001) recovers Diamond with a range of elasticity of input substitution such that $\sigma \geq 1$. In order to clarify further, let us take logarithmic formulations for the utility functions, i.e., $\varepsilon = -1$ with a Cobb-Douglas technology, $\sigma = 1$. Therefore, the eigenvalue (27) is simplified to $J = \alpha \in (0,1)$, thus a unique-path steady state.

6.2 Our model

6.2.1 Separable Preferences

As before, it is assumed that agents have the same utility in both periods, i.e., $\varepsilon=\varepsilon_{11}=\varepsilon_{22}$. The eigenvalue (22) is simplified to:

$$J = \frac{-\alpha\frac{\varepsilon}{\gamma}}{\sigma\left(\eta_2 - \varepsilon - \eta_1\frac{1-\gamma}{\gamma}\varepsilon\right) + (1-\alpha)(1+\varepsilon)} \tag{28}$$

The necessary condition for the existence of cycles of period two is $1+\varepsilon < 0$. Along with (28), the numerator is positive and the SOCs state that $\eta_2 - \varepsilon - \eta\frac{1-\gamma}{\gamma}\varepsilon > 0$. Flip cycles arise at $\sigma = \sigma^F_S$ where

$$\sigma_S^F \equiv \frac{\alpha\frac{\varepsilon}{\gamma} - (1-\alpha)(1+\varepsilon)}{\eta_2 - \varepsilon - \eta_1 \frac{1-\gamma}{\gamma}\varepsilon} \tag{29}$$

Note that cycles appear for the same conditions as in the Benchmark model without costs, i.e., for $\varepsilon < \min(-1, \varepsilon^{sb})$ and $\gamma > \gamma^{sb}$. This shows that transaction costs do not affect the appearance of these cycles, which means that the SOCs dominate the effect of transaction costs.

6.2.2 Non-separable preferences

In the non-separable case, Proposition 4 summarises the conditions under which cycles of period two appear. We only focus on case 1 of Proposition 4, since the other cases have similar intuition.

Suppose that, at period *t*, capital stock K_t increases from its value of the steady state which augments the wage w_t and induces more capital accumulation. The presence of high ε_{11} and γ and small ε_{22} enforces agents to consume more today and to reduce their capital accumulation and so future consumption. This gives rise to two subcases:[17]

On the one hand, whenever $\varepsilon_{12} > 0$ this acts as an offset effect because a reduction in future consumption decreases the marginal utility from first-period consumption and thus leads agents to reduce present consumption. Hence, in order to ensure a reduction in capital accumulation and therefore the emergence of cycles of period two, a sufficiently high sensitivity of costs is required, $\eta_1 > \eta_1^{\ddot{}}$.

On the other hand, whenever $\varepsilon_{12} < 0$, a reduction in future consumption increases marginal utility from the first-period consumption, which provides an additional force to reduce capital accumulation, hence cycles emerge. It is important to notice that in this subcase, cycles appear without any restriction on transaction costs.

In the numerical example next, we clarify the effect of transaction costs on stability range in both the separable and non-separable cases with isoelastic transaction costs formulations.

7. Numerical example

In this section, we confirm numerically our theoretical results presented in the previous section. Let us consider a CES production function $f(a) = A\left[\alpha a^{-\chi} + (1-\alpha)\right]^{-1/\chi}$ with $\alpha \in (0,1), A > 0, \chi > -1\ and \chi \neq 0$ and an isoelastic cost function: $\varphi(s) = s^{\eta}$ with $\eta \in (0,1)$[18]. The annual ratio of personal consumption expenditures over GDP has an average of 0.65 over the period (1959 - 2008) for the US economy. Our objective is to determine the critical values under which flip cycles appear, i.e. $\sigma^F > 0$.

7.1 Non-separable preferences

As in Venditti (2003), we consider the following utility function:

$$u(c,d) = \frac{1}{\theta}\left[\varsigma c^{-\rho} + (1-\varsigma)\, d^{-\rho}\right]^{-\frac{\theta}{\rho}} \tag{30}$$

with $\theta \leq 1, \varsigma \in (0,1)$ and $\rho > -1$, the discount factor is $(1-\varsigma)/\varsigma$ and the elasticity of intertemporal substitution is $1 / (1+\rho)$ and $u_{12} < 0$ if and only if $\rho + \theta$. Given the above technology and the cost function, we get: $c = A(1-\alpha) - (1+n)^{\eta}$ and $d = A\alpha(1+n)$. Using the above consumption values then the steady state value of A can be obtained implicitly using condition (31).

$$\frac{\varsigma}{1-\varsigma}\left(A(1-\alpha) - (1+n)^{\eta}\right)^{-\rho-1}\eta - (1+n)^{-\rho-\eta} A^{-\rho}\alpha^{-\rho} = 0 \tag{31}$$

The positivity of first-period consumption requires that:

$$A > \frac{(1+n)^{\eta}}{1-\alpha} \equiv \hat{A} \tag{32}$$

The steady state value of γ can be obtained endogenously using:

$$\gamma = 1 - \frac{(1+n)^{\eta}}{A(1-\alpha)} \tag{33}$$

One can show directly that the elasticities of preferences as:

$$\begin{aligned} \varepsilon_{11} &= (\theta+\rho)\frac{\varsigma c^{-\rho}}{\varsigma c^{-\rho} + (1-\varsigma)\, d^{-\rho}} - (\rho+1) \\ \varepsilon_{22} &= (\theta+\rho)\frac{(1-\varsigma)\, d^{-\rho}}{\varsigma c^{-\rho} + (1-\varsigma)\, d^{-\rho}} - (\rho+1) \\ \varepsilon_{12} &= (\theta+\rho)\frac{(1-\varsigma)\, d^{-\rho}}{\varsigma c^{-\rho} + (1-\varsigma)\, d^{-\rho}} \end{aligned} \tag{34}$$

Let us set $\alpha = 0.33$, $\eta = 0.5$, $n = 0.5175$, $\rho = 7$, and $\theta = 0.5$, then using (31), we obtain $A = 4.1816 > 1.6915 = \hat{A}$. Given A, we obtain $\gamma = 0.59550$. Given these values, we get $\varepsilon_{12} = 1.2697$, $\varepsilon_{11} = -1.7697$ and $\varepsilon_{22} = -6.7303$. The SOCs is verified as well $2\varepsilon_{12} - \varepsilon_{22} - \eta\frac{1-\gamma}{\gamma}\varepsilon_{11} + \eta - 1 = 8.9303 > 0$ and finally $\sigma^{F} = 0.02872 > 0$.

Table 1

η	0.3	0.5	0.7	0.9
σ	0.02872	4.9344×10^{-2}	5.8739×10^{-2}	6.3552×10^{-2}

From Table 1, we observe that $\partial\sigma^F/\partial\eta > 0$ and $\varepsilon_{12} > 0$, so $\partial J/\partial\sigma < 0$. Notice that the basic model without costs is recovered by setting $\eta = 1$. Hence, the further one is from 1, the higher the sensitivity of costs. As a result, transaction costs act as a destabilising factor, in the sense that they widen the range of parameters giving rise to cycles of period two, that is, $\sigma^F, +\infty$.

7.2 Separable preferences

Suppose that agents have the same utility in both periods with CIES prefer ences: $v(x) = \dfrac{x^{1-\delta}}{1-\delta}$ *with* $\delta > 0$ *and* c, d.

One can easily show that the elasticity of marginal utility in consumption ε = -δ. As before, the steady state value of *A* can be obtained using (16) together with the above cost and production functions. Explicitly, we get:

$$\left(A(1-\alpha) - (1+n)^{\eta}\right)^{-\delta} \eta - \beta\alpha^{1-\delta}(1+n)^{1-\eta-\delta} A^{1-\delta} = 0 \tag{36}$$

where *A* is restricted to the positivity condition of first-period consumption (32). Additionally, the share of first-period consumption is obtained by (33).

Let us set α=0.33, δ=4.44, n=0.5175 and β=0.3. Therefore, ε=-4.44<-1.9706 ≡ ε^{sb}. As a result, we get Table 2:

Table 2

η	0.3	0.5	0.7	0.9
σ_S^F	2.6044 x 10^{-6}	3.750 x 10^{-2}	0.05233	6.1279 X 10^{-2}

Transaction costs influence the stability region through their effects on σ_S^F . Consequently, one can easily show that $\partial\sigma_S^F/\partial\eta > 0$ and, from (28), we get $\partial J/\partial\sigma < 0$. Similar to the previous explanation, transaction costs act as a destabilising factor. For all values of η, the steady state value of *A* (32) is verified, as well as the SOCs.

8. Conclusion

This paper analyses economic stability in an overlapping generations model with exogenous labour supply. We extend the standard one-dimension OLG by introducing transaction costs related to the amount of investment. Young agents consume and save according to wage income while. in the next period, old agents who are retired consume all their saving returns. We consider primarily two different aspects with respect to household preferences. Initially, we focus on a general non-separable formulation of preferences, then the case of separable preferences. It is shown that the presence of transaction costs with respect to saving promotes cycles and it is proved that these costs act as a destabilising factor. It is also demonstrated that under specific conditions, the steady state may be dynamically inefficient (or efficient) if there is an over-accumulation (or under-accumulation) of capital with respect to the Golden Rule, i.e. the net return of capital is higher (or lower) than the population growth. Comparing to the model of Diamond (1965), where he proposes a high substitutability between current and future consumptions, the main contribution of this paper is the emergence of cycles of period two. This paper generalises the stability condition of steady state equilibrium obtained by Diamond.

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Appendix

(A) Sufficient conditions for utility maximisation

Using the Lagrangian function (4), we calculate the associated Hessian matrix with respect to λ_t, μ_τ, c_t, d_{t+1} and s_t: [19]

$$H \equiv \begin{bmatrix} 0 & 0 & -1 & 0 & -\varphi' \\ 0 & 0 & 0 & -1 & r \\ -1 & 0 & u_{11} & u_{12} & 0 \\ 0 & -1 & u_{12} & u_{22} & 0 \\ -\varphi' & r & 0 & 0 & -\lambda_1\varphi'' \end{bmatrix}$$

The household problem is considered as a maximisation problem if and only if the determinant of the leading principal minors of the above Hessian matrix change its sign. In other words, if the determinant of H has the same sign as $(-1)^n$ and the last n-m diagonal principal minors have alternative signs. Here, the number of variables n=3 and the number of constraints m=2. Thus, the optimum is a local maximum only if *det* $H<0$. We need to find the conditions under which the matrix H is negative definite (negative semi-definite) over the set of values satisfying the first-order conditions and the constraints. Therefore:

$$Det\ H = r^2u_{22} - 2r\varphi' u_{12} - \lambda_1\varphi'' + (\varphi')^2 u_{11} < 0$$

Using (8) and the FOCs (5), (6) and (7), we obtain a lower bound for the elasticity of transaction costs with respect to savings:

$$\eta_1 \frac{1-\gamma}{\gamma}\varepsilon_{11} - 2\varepsilon_{12} + \varepsilon_{22} < \eta_2$$

(B) Proof of proposition 3

Proof. According to definitions 1 - 3 and Assumptions 6 and 7, over-accumulation of capital gives $Af'(a^*)/\varphi'[(1+n)a^*]<1+n$. This demonstration is based on previous work of De la Croix and Michel (2002) and Drugeon *et al* (2010). We have to prove that we can decrease capital stock and raise consumption at one date without reducing consumption at another date. In a neighborhood $(a^* - 2\varkappa, a^* + 2\varkappa)$ of a^*, we have $Af'(a)/\varphi'[(1+n)a] < 1+n$. After some date t_0, we have $a_t \in (a^* - 2\varkappa, a^* + 2\varkappa)$ with $Af'(a_t)/\varphi'[(1+n)a_{t+1}] < 1+n$ and $Af'(a_t - \varkappa)/\varphi'[(1+n)(a_{t+1} - \varkappa)] < 1+n$. The concavity of $f(.)$ and $\varphi[.]$ implies respectively:

$$Af(a - \varkappa) - Af(a) \geq -Af'(a - \varkappa)x$$

and

$$\varphi[(1+n)(a - \varkappa)] - \varphi[(1+n)a] \geq -\varphi'[(1+n)(a - \varkappa)](1+n)\varkappa$$

Let us decline capital stock by $\varkappa$ after date t_0 and forever. Investment a_{t_0+1} is reduced by $\varkappa$ and consumption $\Sigma(a_{t_0}, a_{t_0+1})$ is increased by $\varphi'[(1+n)a_{t_0+1}](1+n)\varkappa$. At date $t > t_0$, the new consumption level is:

$$\begin{aligned}
& \Sigma(a_t - \varkappa, a_{t+1} - \varkappa) \\
= \; & Af(a_t - \varkappa) - \varphi[(1+n)(a_{t+1} - \varkappa)] \\
\geq \; & Af(a_t) - Af'(a_t - \varkappa)\varkappa - [\varphi[(1+n)a_{t+1}] - \varphi'[(1+n)(a_{t+1} - \varkappa)](1+n)\varkappa] \\
\geq \; & Af(a_t) - \varphi[(1+n)a_{t+1}] + [\varphi'[(1+n)(a_{t+1} - \varkappa)](1+n) - Af'(a_t - \varkappa)]\varkappa \\
> \; & Af(a_t) - \varphi[(1+n)a_{t+1}] = \Sigma(a_t, a_{t+1})
\end{aligned}$$

So, consumption can be increased for all future periods and the path is dynamically inefficient. Now, we go forward to show that whenever $Af'(a^*)/\varphi'[(1+n)a^*] > 1+n$, then there exists an under-accumulation of capital. To prove this, it is enough to show the impossibility of raising one period t_1 consumption without reducing other period's consumption. Moreover, $Af'(a^*)/\varphi'[(1+n)a^*] > 1+n$ gives that $Af'(a^*)/\varphi'[(1+n)a^*] > b(1+n)$ with some $b > 1$. Along an equilibrium path and for $t \geq t_0$, we have $Af'(a_t)/\varphi'[(1+n)a_t] > b(1+n)$. At any date t, the difference from another feasible path $\bar{a}_t$ satisfies:

$$\begin{aligned}
\Delta C_t &= Af(\bar{a}_t) - Af(a_t) - (\varphi[(1+n)\bar{a}_{t+1}] - \varphi[(1+n)a_{t+1}]) \\
&\leq Af'(a_t)(\bar{a}_t - a_t) - \varphi'[(1+n)a_{t+1}](1+n)(\bar{a}_{t+1} - a_{t+1})
\end{aligned}$$

where ΔC_t is the difference of total consumption. This implies:

$$\varphi'[(1+n)a_{t+1}](1+n)(\bar{a}_{t+1} - a_{t+1}) \leq Af'(a_t)(\bar{a}_t - a_t) - \Delta C_t \tag{38}$$

Assume that consumption never decreases, which means that capital never increases. Indeed, by induction if $\overline{a}_t - a_t \leq 0$, which is true at t=0, and if $\Delta C_t \geq 0$, then (40) implies $\overline{a_{t+1}} - a_{t+1} \leq 0$. Moreover, suppose that consumption increases at time t_1: $\Delta C_{t1} > 0$, then the previous argument gives $\overline{a}_t - a_t < 0$ for all $t > t_1$. This implies that for $t > t_2 = \max(t_0, t_1)$:

$$\begin{aligned}(1+n)\left(\bar{a}_{t+1} - a_{t+1}\right) &\leq \frac{Af'\left(a_t\right)}{\varphi'\left[(1+n)\,a_{t+1}\right]}\left(\bar{a}_t - a_t\right) \\ &< b\,(1+n)\left(\bar{a}_t - a_t\right)\end{aligned}$$

since $\bar{a}_t - a_t < 0$ and $Af'\left(a_t\right)/\varphi'\left[(1+n)\,a_{t+1}\right] > b\,(1+n)$. Hence,

$$\bar{a}_{t+1} - a_{t+1} < b\left(\bar{a}_t - a_t\right)$$

and:

$$\overline{a_{t+1}} - a_{t+1} < b^{t-t_2}\left(\overline{a}_t - a_t\right) < 0$$

as $b > 1$ and a_{t+1} converge to the steady state, we have $\overline{a}_t - a_t$ converge to $-\infty$ and $\overline{a_{t+1}}$ becomes negative which is impossible.

(C) Proof of Proposition 4

In this proof, we show the existence of flip bifurcation under different configurations. As shown before, the flip cycles appear at $\sigma = \sigma^F$ given by (23) and since its denominator is positive, the existence of flip bifurcation requires a negative numerator, that is:

$$\eta_1\left(\frac{\varepsilon_{11}}{\gamma} - \frac{1-\alpha}{\alpha}(1-\varepsilon_{12}+\varepsilon_{22})\right) > \frac{\varepsilon_{12}}{1-\gamma} \tag{39}$$

In order to simplify, let us take two different cases concerning the sign of ε_{12}.

1. $\varepsilon_{12} > 0$

In (39), the sign of $\varepsilon_{11}/\gamma - (1-\alpha)(1-\varepsilon_{12}+\varepsilon_{22})/\alpha$ is unknown. However, condition (39) cannot hold whenever $\varepsilon_{11}/\gamma - (1-\alpha)(1-\varepsilon_{12}+\varepsilon_{22})/\alpha < 0$. Thus, $\varepsilon_{11}/\gamma - (1-\alpha)(1-\varepsilon_{12}+\varepsilon_{22})/\alpha > 0$ is a necessary condition in order for condition (39) to verify and thus $\sigma^F > 0$ is satisfied for $\eta_1 > \eta_1{}^*$.

Moreover, $\varepsilon_{11}/\gamma - (1-\alpha)(1-\varepsilon_{12}+\varepsilon_{22})/\alpha > 0$ requires $\varepsilon_{22} < \varepsilon^*_{22}$ and can be written as $\gamma \geq \gamma^*$. Since γ represents the consumption share out of wage, that is, $\gamma \in (0,1)$, therefore $\gamma^* < 1$ if and only if $\varepsilon_{11} > \varepsilon^*_{11}$.

As a result, flip bifurcation occurs at $\sigma = \sigma^F$ whenever $\varepsilon_{11} > \varepsilon^*_{11}$, $\gamma > \gamma^*$, $\varepsilon_{22} < \varepsilon^*_{22}$ and $\varepsilon_{12} > 0$ (condition (1.*i*)).

2. $\varepsilon_{12} < 0$.

We consider the following configurations:

A. $\varepsilon_{11}/\gamma - (1-\alpha)(1-\varepsilon_{12}+\varepsilon_{22})/\alpha > 0$

As before, $\sigma^F > 0$ for all $\eta_1 > 0$. Condition A requires $\varepsilon_{22} < \varepsilon^*_{22}$ and is equivalent to $\gamma > \gamma^*$ However, $\gamma^* < 1$ for $\varepsilon_{11} > \varepsilon^*_{11}$. This implies condition (1.*ii*).

B. $\varepsilon_{11}/\gamma-(1-\alpha)(1-\varepsilon_{12}+\varepsilon_{22})/\alpha<0$

In this case, $\sigma^F>0$ if and only if $\eta_1>\eta_1^*$. For $\varepsilon_{22}>\varepsilon^*_{22}$, then condition (B) is verified for all $\gamma\in(0,1)$. However, for $\varepsilon_{22}<\varepsilon^*_{22}$, condition (B) is equivalent to $\gamma<\gamma^*$. In this case, $\gamma^*<1$ if and only if $\varepsilon_{11}>\varepsilon^*_{11}$ (conditions (2) and (3)).

Endnotes

1. University of Birzeit, Economic Department, maburjaile@birzeit.edu.

2. See among others, Alan (2006), Lo et al (2004) and Vissing-Jorgensen (2002).

3. Geanakoplos and Polemarchakis (1986), Reichlin (1986) and Cazzavillan (2001) assume that agents consume only in the second-period.

4. Please refer to the brokerage commission and fee schedule, Fidelity Investments (2013).

5. As in standard two-period OLG models, we assume a full depreciation rate that is $N_t s_t = K_{(t+1)}$.

6. Note that $u_1(c_t, d_{t+1}) \in \partial u(c_t, d_{t+1})/\partial c_t$ is the partial derivative with respect to the first variable in the utility function, while $u_2(c_t, d_{t+1}) \in \partial u(c_t, d_{t+1})/\partial d_{t+1}$ is the first derivative with respect to the second variable in the utility function.

7. In this environment, considering flat or fixed transaction costs are not helpful to capture its effects on dynamic stability.

8. Since $\xi(s_t)$ is increasing, the function $\varphi(s_t)$ has the same properties of $\xi(s_t)$ as mentioned in Assumption 2.

9. We denote $\gamma\equiv c/A\omega(a)$ as the share of first-period consumption over wage income.

10. Similar to de la Croix and Michel (2002), we define net production as production minus investment and its associated transaction costs.

11. If this assumption is violated, we cannot determine the stationary capital-labour ratio that maximises net production.

12. The term 'Golden Rule' was introduced by Phelps (1961).

13. $\Sigma(a)$ is defined in (18).

14. See Appendix (A).

15. Diamond finds that the steady state exhibits saddle-path stability if and only if the elasticity of saving with respect to interest is not negative, which means a high elasticity of intertemporal substitution in consumption. Simply, in this model, high substitutability implies $1+\varepsilon>0$.

16. Cazzavillan and Pintus (2004) find that endogenous fluctuations require $\sigma>\alpha$.

17. If the utility function exhibits homogeneity of degree one, then $\varepsilon_{12}>0$. As a result, the conditions under which two-period cycles appear are summarised in Case (1) of Proposition (4).

18. Consistent with previous notations together with an isoelastic cost function $\phi(s)=s^{\eta}$, we obtain $\eta_1=\eta$ and $\eta_2=\eta$-1.

19. For simplicity, we omit the arguments and the time subscripts related with the functions.

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