

A Complete Characterization of Economies with the Nonsubstitution Property

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ABSTRACT

A complete characterization of economies with the nonsubstitution property is presented for linear models. In this characterization, a degree of proper joint production as well as the existence of durable capital goods is allowed for.

1. INTRODUCTION

In Schefold(1978a), it is shown that a constant returns to scale economy with joint production can have the properties of one with single production, if the efficient set of processes forms a square matrix of net products and it has its nonnegative inverse. (See also Schefold 1978b). Then Herrero and Villar (1988) proved this very property is a necessary and sufficient condition for an economy to enjoy the nonsubstitution property due to Georgescu-Roegen (1951) and Samuelson (1951). This proposition was in fact obtained more than two decades ago by Dasgupta and Sinha (1979), but its publication was delayed until 1992 (see Dasgupta, 1992 and Dasgupta and Sinha, 1992). Then, Bidard (1991) presented a necessary and sufficient condition for a real square matrix to have a strictly positive inverse. Following this, Erreygers (1996) rediscovered propositions due to Dasgupta (1992), and Bidard and Erreygers (1998) gave a necessary and sufficient condition for a square matrix to have a nonnegative inverse.

In the above contributions, though a certain class of joint production is admitted, an important group of joint products, i.e., durable capital goods, are in general excluded because the *net output square matrix composed of efficient processes normally does not have a nonnegative inverse*. In connection with the nonsubstitution theorem, the existence of durable capital goods is dealt with in Kurz and Salvadori (1995, Chapters 7 and 9). Their method is to require that, in the final demand vector, the elements corresponding to old durable capital goods be zero. Using this method, Fujimoto, Silva and Villar (2002a) obtained a necessary and sufficient condition under which a given real

square matrix has an inverse whose columns in a particular subset of indices are nonnegative: that subset of indices for commodities represents perishable goods as well as brand new durable capital goods.

In this article, we combine the result by Dasgupta (1992), the method in Kurz and Salvadori (1995) explained above, and a mathematical theorem in Fujimoto, Silva and Villar (2002) to obtain a full characterization of linear economic models in which the nonsubstitution theorem holds good. In section 2, we explain our model and the nonsubstitution theorem with its proof. The method of proof is based on duality in linear programming, and is due to Chander (1974). Section 3 includes a numerical example to illustrate our results. In the final section, several remarks are given.

For more recent contributions, see also Kuga (2001), Hasfura-Buenaga, Holder and Stuart (2002), and Villar (2002). For inverse positive matrices related to M -matrices, the reader is referred to Berman and Plemmons (1979).

2. LINEAR MODELS WITH THE NONSUBSTITUTION PROPERTY

Let us consider an ordinary linear input-output model with joint production and durable capital goods. First we explain our notation. The symbol \mathbb{R}^n means the real Euclidean space of dimension n ($n \geq 2$), and \mathbb{R}_+^n the nonnegative orthant of \mathbb{R}^n . The two given real $m \times n$ nonnegative matrices A and B map from \mathbb{R}_+^n into \mathbb{R}_+^m . One more symbol ℓ^n is the row vector in \mathbb{R}^n whose entries are all unity.

The inequality signs for vector comparison are as follows:

$$\begin{aligned} \mathbf{x} \geq \mathbf{y} & \text{ iff } \mathbf{x} - \mathbf{y} \in \mathbb{R}_+^n; \\ \mathbf{x} > \mathbf{y} & \text{ iff } \mathbf{x} - \mathbf{y} \in \mathbb{R}_+^n - \{0\}; \\ \mathbf{x} \square \mathbf{y} & \text{ iff } \mathbf{x} - \mathbf{y} \in \text{int}(\mathbb{R}_+^n), \end{aligned}$$

Now in our linear economic model, the matrix A is understood as the input coefficient matrix, and the matrix B as the output coefficient matrix. We denote $B - A$ as M . The number m is that of commodities, while n represents the number of processes available. We assume $m \leq n$. This assumption may be acceptable when we take into consideration those processes each of which uses one particular commodity as input and keeps it for one year using a certain amount of labour. The index set for commodities, C , is defined as $C \equiv \{1, 2, \dots, m\}$. The processes are normalized so that the labour input coefficient is unity in each process, implying that every process needs the sole primary input labour. Thus, the n -row vector $\ell^n \equiv (1, 1, \dots, 1)$ stands for the labour input coefficient vector. A column vector $\mathbf{x} \in \mathbb{R}_+^n$ shows a corresponding activity vector with its j -th entry x_j meaning the activity level of process j . Then $A\mathbf{x}$ stands for the material input vector, and $\ell^n \cdot \mathbf{x}$ for the labour input, while $B\mathbf{x}$ shows the material output vector. The vector, $B\mathbf{x} - A\mathbf{x}$, then, represents the net output vector. The symbol $(A\mathbf{x})_i$ means the i -th element of $(A\mathbf{x})$.

In our model, we allow for proper joint production and durable capital goods. Thus, a certain set of columns of B may include more than one positive entry. We classify goods into two categories, C_1 and C_2 : the first category, C_1 , consists of consumption goods and brand new capital goods, which can appear in a final demand vector as positive entries, and the second, C_2 , of old durable capital goods, which do not appear in any final demand vector. Final demands are to be understood as either for consumption or for exportation. Demand for capital accumulation is included in the term $g \cdot Ax$, where g is a rate of steady balanced growth: this growth rate is assumed to be zero for simplicity in this article. Naturally, we have $C = C_1 \cup C_2$.

We first solve in this model the following linear programming problem:

$$\left. \begin{array}{l} \min \ell^n \cdot \mathbf{x} \\ \text{s.t.: } B\mathbf{x} \geq A\mathbf{x} + \mathbf{d}^o \\ \mathbf{x} \in \mathbb{R}_+^n \end{array} \right\} \quad (1)$$

where $\mathbf{d}^o \geq 0$ is a given final demand vector such that $d_i^o > 0$ for $i \in C_1$, and $d_i^o > 0$ for $i \in C_2$.

We assume:

Assumption A1. There exists an $\mathbf{x} \in \mathbb{R}_+^n$ such that $(B\mathbf{x})_i > (A\mathbf{x})_i$ for $i \in C_1$, and $(B\mathbf{x})_i \geq (A\mathbf{x})_i$ for $i \in C_2$.

Assumption A1 is a standard productivity assumption, applied to production models with fixed capital, and guarantees the existence of an optimal solution to the problem (1). Then, we assume:

Assumption A2. An optimum solution \mathbf{x}^o to the problem (1) has at least m positive entries.

Assumption A2 says that any efficient choice of techniques must involve at least m processes. This is a necessary condition for a full nonsubstitution theorem to hold, as it ensures that the efficient techniques may be able to span the whole commodity space.

Let \mathbf{x}^o have at least m positive entries as is guaranteed by Assumption A2. Now create two matrices B^o and A^o by collecting those processes (i.e., columns) which are actually used in \mathbf{x}^o , and define $M^o \equiv B^o - A^o$. Note that M^o is a matrix with m rows and k columns, $n \geq k \geq m$. Further assume:

Assumption A3. If $(M^o\mathbf{x})_i > 0$ for $i \in C_1$, and $(M^o\mathbf{x})_i = 0$ for $i \in C_2$, then there are at least m positive entries in \mathbf{x} .

Assumption A3 is a weak form of inverse isotoneity. It requires that efficient production of strictly positive amounts of all brand new commodities should utilize at least m processes.

Remark 1 Note that vector \mathbf{x}° in Assumption A2 is a solution to (1) which determines M° , whereas vector \mathbf{x} in Assumption A3 is an arbitrary vector so long as it satisfies the conditions therein.

We are ready to prove:

Lemma 1. Given Assumptions A1, A2, and A3, problem (1) has an optimal solution \mathbf{x}^* which has exactly m positive elements.

Proof. By A2, \mathbf{x}° has at least m positive elements. Since there can be at most m independent columns in M° , we can remove columns one by one from M° while keeping an optimal solution nonnegative, until we find an \mathbf{x}^* . \square

As above, we create two matrices B^* and A^* by collecting those processes which are used in \mathbf{x}^* , and define $M^* \equiv B^* - A^*$. This matrix M^* is $m \times m$, that is, square.

Lemma 2. Given Assumptions A1, A2, and A3, the matrix M^* is regular, and the inverse of (M^*) has nonnegative columns when their index belongs to C_1 .

Proof. This is a result in Fujimoto, Silva and Villar (2002a), and the proof is omitted. \square

Here is our main result.

Theorem 1. Given Assumptions A1, A2, and A3, when $B\mathbf{y} \geq A\mathbf{y} + \mathbf{d}$ for some $\mathbf{y} \in \mathbb{R}_+^n$ and an arbitrary $\mathbf{d} \in \mathbb{R}_+^m$ such that $\mathbf{d}_i \geq 0$ for $i \in C_1$, and $\mathbf{d}_i = 0$ for $i \in C_2$, a solution \mathbf{y}^* to $B^*\mathbf{y}^* = A^*\mathbf{y}^* + \mathbf{d}$ satisfies $\ell^m \cdot \mathbf{y}^* \leq \ell^m \cdot \mathbf{y}$.

Proof. Following Chander(1974), let us consider the dual program to (1):

$$\left. \begin{array}{l} \max \mathbf{p} \cdot \mathbf{d} \\ \text{s.t. : } \mathbf{p}B \leq \mathbf{p}A + \ell^m \\ \mathbf{p} \in \mathbb{R}_+^m \end{array} \right\} \quad [2]$$

An optimal solution to [2], $\mathbf{p}^* \geq 0$, can be obtained by solving the equation $\mathbf{p}^*B^* = \mathbf{p}^*A^* + \ell^m$. It should be noted that $\mathbf{y}^* \in \mathbb{R}_+^m$ because of Lemma 2 and the property supposed for \mathbf{d} . Then, postmultiply by \mathbf{y}^* both sides of this equation, and we get

$$\mathbf{p}^*B^*\mathbf{y}^* = \mathbf{p}^*A^*\mathbf{y}^* + \ell^m \cdot \mathbf{y}^* \quad (3)$$

From $B^*\mathbf{y}^* = A^*\mathbf{y}^* + \mathbf{d}$, it follows that

$$\mathbf{p}^*B^*\mathbf{y}^* = \mathbf{p}^*A^*\mathbf{y}^* + \mathbf{p}^* \cdot \mathbf{d} \quad (4)$$

through premultiplying by \mathbf{p}^* . From eqs. (3) and (4), we have

$$\ell^m \cdot \mathbf{y}^* = \mathbf{p}^* \cdot \mathbf{d} \quad (5)$$

On the other hand, from $B\mathbf{y} \geq A\mathbf{y} + \mathbf{d}$, premultiplying both sides by \mathbf{p}^* yields

$$\mathbf{p}^* B \mathbf{y} \geq \mathbf{p}^* A \mathbf{y} + \mathbf{p}^* \cdot \mathbf{d} \quad (6)$$

while from the constraint $\mathbf{p}^* B \leq \mathbf{p}^* A + \ell^n$ we have

$$\mathbf{p}^* B \cdot \mathbf{y} \leq \mathbf{p}^* A \mathbf{y} + \ell^n \cdot \mathbf{y} \quad (7)$$

From inequalities (6) and (7), it follows that $\mathbf{p}^* \cdot \mathbf{d} \leq \ell^n \cdot \mathbf{y}$ (8)

Therefore, [5] and [8] show that our Theorem is valid. \square

3. A NUMERICAL EXAMPLE

Consider the case in which there are two perishable goods which serve both for consumption and for production, and one durable capital good (or a machine), indexed as 1, 2, and 3 respectively. A machine lasts for two years, and is discarded without any scrap value. One year old machine is indexed as 4. There are five basic processes described as follows.

$$B \equiv \begin{pmatrix} 1 & 1 & 0 & 1 & 0.2 \\ 1 & 1 & 1 & 0 & 0.2 \\ 0 & 0 & 1 & 1 & 0.2 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, A \equiv \begin{pmatrix} 0 & 0.2 & 2 & 0 & 0 \\ 0 & 0.1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \ell^s \equiv (1,1,1,1,1), \text{ and}$$

$$M \equiv \begin{pmatrix} 1 & 0.8 & -2 & 1 & 0.2 \\ 1 & 0.9 & 1 & -2 & 0.2 \\ -1 & 0 & 1 & 1 & 0.2 \\ 1 & -1 & 0 & 0 & 0 \end{pmatrix}$$

The two categories of commodities are $C_1 \equiv \{1, 2, 3\}$ and $C_2 \equiv \{4\}$. Note that commodity 3 is brand new, while commodity 4 is one year old. And its depreciation is not quantitative, i.e., there is no simple way to convert a unit of one year old machine to a certain amount of a brand new one. Moreover, in every process, proper joint production is involved, and yet it is not difficult to see that our assumptions are satisfied.

The fifth process turns out to be inefficient, and the first four processes form the optimal basis M^* , and its inverse is

$$(M^*)^{-1} = \begin{pmatrix} 1 & 0.8 & -2 & 1 \\ 1 & 0.9 & 1 & -2 \\ -1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0.370\dots & 0.370\dots & 0.370\dots & 0.630\dots \\ 0.370\dots & 0.370\dots & 0.370\dots & -0.370\dots \\ 0.012\dots & 0.346\dots & 0.679\dots & 0.321\dots \\ 0.358\dots & 0.025\dots & 0.691\dots & 0.309\dots \end{pmatrix}$$

Since the inverse of M^* includes a negative entry, this example cannot be covered by the results in Herrero and Villar (1988) and Dasgupta (1992). The equilibrium price vector, $\mathbf{p}^* = (1.111\dots, 1.111\dots, 2.111\dots, 0.889\dots)$, which confirms that the fifth process is inefficient, making losses at these prices.

The first four processes are modified from an example in Dasgupta (1992), which is due to Sinha: that is,

$$\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix} .$$

This matrix is inverse-positive.

4. REMARKS

Our way of making assumptions step by step as we solve a particular programming problem makes it possible to allow for the existence of those processes which need no material input, which turn up as inefficient. This is illustrated by the fifth process in the numerical example.

If Assumption A3 is not satisfied with an index set smaller than C_2 (and an index set larger than C_1), the columns in the index set C_2 of the inverse $(M^*)^{-1}$ contain at least one negative entry. This is also illustrated in the fourth column of $(M^*)^{-1}$.

Although it has been required that $d_i = 0$ for $i \in C_2$, in fact, they can be positive to some extent. With respect to the equation $M^* \mathbf{x} = \mathbf{d}$, in order for \mathbf{x} to remain nonnegative, d_i for $i \in C_2$ should be bounded from above provided d_i for $i \in C_1$ are given, because the columns of the inverse $(M^*)^{-1}$ in the index set C_2 contain negative entries.

Our characterization is 'complete' in the sense that linear models with joint production and durable capital goods should satisfy Assumptions A1, A2, and A3 when those models are required to have the nonsubstitution property and every final demand vector, where entries for old machines are zero, is produced with no slack. The condition of nonnegative invertibility in Herrero and Villar (1988) is 'necessary and sufficient' for the nonsubstitution theorem to hold, only when models can produce any strictly positive final demand vectors, including aged capital goods.

Having presented a complete characterization of the nonsubstitution theorem, it is then possible to extend the model to one in which synergetic external economies or diseconomies operate as is done in Fujimoto, Silva and Villar (2002b).

Finally, since some kinds of proper joint production and qualitative depreciation are allowed for, we cannot, in establishing the nonsubstitution theo-

rem, use the methods of proof based on cost functions, which are employed in Morishima (1964), Stiglitz (1970), Johansen (1972) (as amended by Dasgupta (1974)), Fujimoto (1980), Fujimoto (1987), and Kuga (2001).

Accepted for publication: 10 April 2003

ENDNOTE

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