
New Methods and Understanding in Economic Dynamics? An Introductory Guide to Chaos and Economics

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Abstract

This paper considers the relevance of chaos theory to economics. It provides an introductory survey of the central ideas of chaos and non-linear dynamical systems at a level which is appropriate for the non-technical reader, and provides pointers to accessible reading. The failures of linear modelling are outlined, after which the major features of non-linear and chaotic dynamical systems are explained and explored. This is followed by sections discussing the implications for economic methodology and the directions of theoretical and empirical research.

1. Introduction

Mankind has often sought to discern order within the world. The planets have been found to obey laws which are summarised by equations, making possible the prediction of all future and previous movements. Other phenomena such as weather patterns and turbulence display less regular behaviour which cannot be encapsulated by classical deterministic equations. It is tempting to ask why some phenomena are governed by stable laws whilst others are apparently not, and further to ask whether these are governed by, as yet, undiscovered laws, or are purely random or embrace a mixture of random and deterministic factors. In mathematics and science chaos theory has brought

understanding of phenomena so irregular that they defy prediction.

So, researchers have attempted to apply the concepts to gain understanding of economic activity. Unfortunately, for the general reader much of this burgeoning literature is either too popularistic (e.g. Gleick (1988)), too scientifically and mathematically orientated (e.g. Feigenbaum (1978)), too specialist (e.g. Brock (1986), Medio and Gallo (1993)) or too superficial (e.g. Lipsey (1989)). This paper provides an elementary exposition of some of the main aspects of chaos and its relevance to economics.²

Many phenomena involve periodic or fluctuating behaviour. Examples include planetary motions, weather patterns, biological populations and share price movements. Sometimes these are well behaved, understood, and capable of being represented by simple dynamic models taking the present magnitude of a variable as a function of time and the values of that variable in previous time periods. Such systems are deterministic in the sense that if equations representing them can be discovered and solved, then knowing the state at any point in time provides knowledge of all states future and past. Behaviour is completely determined for all time. The orbiting of the planets is the clearest example of this apparent predictability. In other instances clear patterns are not discernible and activity appears irregular. A glance at the financial page of

any newspaper showing any financial series is all that is needed to confirm this apparent complexity.

Irregular behaviour has proved intractable to modelling. Much effort has been invested in attempts to understand the underlying processes that generate such patterns. Two approaches can be identified. The first assumes that where behaviour is complicated, there are numerous factors influencing it which must be incorporated. This leads to the creation of highly complex forecasting models. The second assumes that there is an underlying trend, but that observed variability is due to the presence of white noise or random shocks which distort and obscure it. Attempts are then made to isolate and identify this trend.

It is usual to *linearise* by making linear approximations, so ensuring that analytical solutions can be obtained, that is to simplify by assuming that there exists a constant proportionality between changes in variables. A property of such models is that *small* changes in independent variables lead to *small* changes in dependant variables. Thus, small measurement errors will not produce large discrepancies in predictions. A major reason for linearising is that many researchers possess a deep rooted belief that the world is governed by natural laws and therefore explainable, provided that those laws can be discovered. This is not only the motivation for research, but conditions its direction and approach, and forms the dominant paradigm, whereby the world is an ordered one. Some remarkable successes have been attained in the physical sciences, but equally many areas have seen less success. There has been little predictive success in fields as diverse as long-range weather forecasting and economic modelling. It is now known that sometimes what was thought to be random or complex behaviour is, instead, chaotic. The term

chaotic is used here in a special sense. Chaotic behaviour is not stochastic or random. On the contrary, a chaotic system is one that is completely deterministic, yet *appears* as if it were purely random, even to the extent of satisfying standard tests of randomness. Such systems are not predictable. Further, they do not necessarily require systems of complex equations to describe them. Remarkably, chaos may be generated from the simplest of *non-linear* equations where, unlike in linear systems, the smallest of changes can lead to extreme variability. Simple causes may produce complex behaviour. So, the observation of complex phenomena does not, of itself, imply complex causes.

Chaos and the related concepts of *fractal geometry* and *complexity* are wide ranging and have a relevance extending far beyond our current discussion. In this essay we restrict ourselves to a consideration of only one aspect of non-linearity and the way it might enhance understanding of the volatility within economic series.

2. Problems of linear modelling

Consider the well known elementary cobweb model. This has provided insights into influences underlying price movements. Market equations are manipulated to obtain a first order *linear difference equation*:

$$P_t = f(P_{t-1}) \quad (1)$$

An analytical solution is derived to identify the time path of *all* prices over *all* time.³ Thus, equation (2) expresses price (P_t) solely as a function of time (t), given some initial price (P_0):

$$P_t = f(t, P_0) \quad (2)$$

The determinate nature of prices is clear. *All* prices are completely determined by the initial

conditions. Knowing the price at any point in time implies that all past and future prices are equally known.

However, this simplifies too much. Predicted price activity never mirrors the complexities of real price movements. Real data can display abrupt changes, which cannot be predicted from linear models. So, there is a resort to *ad hoc* explanation through the invocation of special causes. Part of the problem lies with the assumption of linearity. This implies that P_t always varies in fixed proportion to changes in P_{t-1} , irrespective of the level of P_{t-1} . So, time paths are restricted to exploding, or damped oscillations or a regular 2-cycle. More complex patterns can be generated from more complicated equations, but they still display a regularity not found in real data.

Linear differential equations are often used to model natural phenomena when time is a continuous variable. This is a powerful technique. But, the successfully predicted behaviour of, for example, electrical phenomena, radio waves, etc. displays a regularity and periodicity which is absent from economic data. Even so, this approach breaks down when attempting to model irregular behaviour. For example, it is possible to model fluid flow, but only up to a point. As the speed is increased above some threshold the flow becomes turbulent, the equations break down and the flow pattern becomes unpredictable. While linear differential equations have proved powerful in many areas, they are less useful in modelling irregular phenomena. So it is with economic activity, where additionally it is less appropriate to regard time as continuous, since economic decisions are taken at discrete intervals.

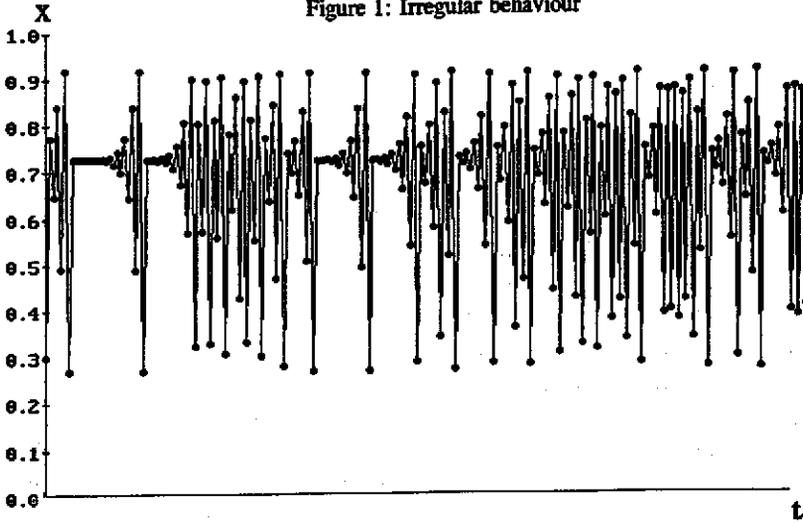
So, deterministic linear models cannot predict, even qualitatively, the sorts of highly

complex price behaviour of real markets. It is often claimed that there are patterns within share price movements, but there seem, also, to be elements of randomness. Few would claim them to be *purely* random movements, and many generations of theorists, empiricists and men in the street have striven to uncover their secret. If they are purely random, then it is a fool's errand, and economists might as well shut up shop and head for the casinos!⁴ The quest continues, but the ability to predict correctly remains elusive. Hence, it is claimed that, some external shock must have thrown the system out of joint, or some important influence has been overlooked. In efforts to simplify, something of importance must have been omitted, yet, clearly all cannot be taken into account. Models, if they are to be of any value, must be simplified representations.

To explain irregularity, is it necessary to introduce ever more complex equations? Is irregularity something that is purely the product of some random process or is there some, as yet, indiscernible underlying order? This raises philosophical issues.⁵ Why should a god ordain the regularity of some phenomena, but apparently take no interest in the behaviour of others? The movements of the planets are regular and ordered. Do other natural phenomena such as weather, earthquakes, natural disasters and economic activity conform to some as yet undiscovered laws, or are they truly random and therefore unpredictable, except upon some basis of the allocation of probabilities?

It is in this sort of dilemma that concepts from chaos theory are helpful. It is now known that extremely complex patterns of behaviour can have very simple causes. The search is on for simpler, rather than more complex, explanations.

Figure 1: Irregular behaviour



3. Non-linear dynamical systems

Figure 1 shows successive values of a variable x . It is difficult to see any pattern, yet surprisingly, in this example the behaviour is completely determined by the simplest of non-linear difference equations. This marked irregularity has some semblance of the sorts of irregularity seen in the world around us.

We now examine the rich patterns of behaviour that are generated by the simple non-linear difference equation:

$$X_t = rX_{t-1}(1 - X_{t-1}/M) \tag{3a}$$

where, in the specific series shown in figure 1, $r = 3.68$, $X_0 = 0.3$ and $M = 1$. This is the logistic equation.⁶ It has been used as a simple model of biological growth where there is some maximum limit to the sustainable population, for example, changing numbers of fish in a pond of given size. The rationale is that, when the numbers are high there are pressures upon resources causing population to decline, whereas when the population is low, resources are relatively abundant and fish can freely multiply. Hence, the population (X_t) in any period (t) depends upon the population (X_{t-1}) in the previous

period, and upon a constant growth parameter ($r > 1$).

M is a constant representing the maximum attainable level of X . It is usual and convenient to define the limit (M) as unity. Thus, actual population values are expressed as fractions of this limit, as in (3b), where lower case x is to be read as a fraction of M (i.e. $x = X/M$):

$$x_t = r.x_{t-1}(1-x_{t-1}) \tag{3b}$$

Removing brackets it is expressed in familiar quadratic form:

$$x_t = r.x_{t-1} - rx_{t-1}^2 \tag{3c}$$

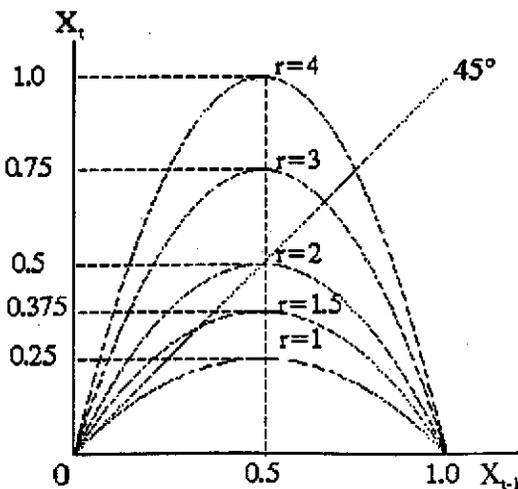
It is obvious that as x increases $(1-x)$ decreases and vice versa. Thus, feedback ensures that the population remains within the limits of zero and unity. So, when the population is low, x increases in the subsequent period, and when the population is high, x decreases. This presence of feedback is of considerable importance in the behaviour of non-linear systems.

The same equation has been employed to provide simple models of economic

phenomena. Savit (1988, 1989) used it as a model for commodity prices. Likewise Baumol and Quandt (1985) took the same equation to provide an elementary model of advertising expenditures, and Jensen and Urban (1984) used it to represent a cobweb model. It is, therefore, clearly possible to think of *real* economic phenomena for which this equation could provide a first approximation. Nevertheless, for the moment we restrict ourselves to an examination of its mathematical properties and the time paths generated.

Equation (3) has roots at $x_{t-1} = 0$ and at $x_{t-1} = 1$, and attains a maximum when $x_{t-1} = 0.5$. Substituting this into (3) gives the maximum: $x_t = 0.25r$ for all r . The general form is quadratic, but the particular form depends upon the value of r . The greater the value of r , the higher the hump, as depicted in figure 2. When $r = 4$, the function attains a maximum of $x_t = 1$. Since x cannot by definition exceed unity, it follows that r cannot exceed 4. Thus $x_t = f(x_{t-1})$ represents a continuous mapping of f into itself for $0 \leq x \leq 1$, i.e. the unit interval.⁷

Figure 2: Particular quadratics



Unlike linear difference equations, (3) cannot be solved analytically and expressed in the form of $x_t = f(t)$. To examine the behaviour of x over time, we are reduced to *graphical* or *iterative* methods. The development of computational power now makes it an easy matter to simulate the behaviour of x^8 , and previously unsuspected patterns have been discovered. However, for pedagogic reasons we initially adopt a graphical approach.

Consider how the qualitative patterns are affected by variations in r . Setting $x_t = x_{t-1}$ in (3) and solving:

$$x_{s1} = 0 \text{ and } x_{s2} = (r-1)/r \quad (4)$$

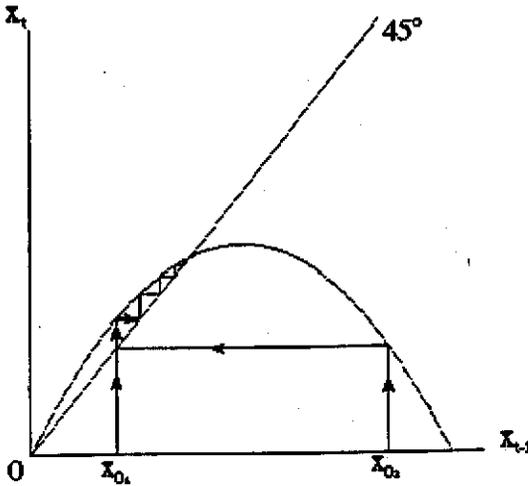
These give the values of x at which the function intersects the 45 degree line. The first of these always occurs at the origin, but the second is dependent upon r . These solutions are *fixed points* or equilibria. Should x attain either value, then that value will persist. Now, when $x = 0$ the first derivative of (3) has a value equalling r , i.e. r gives the slope of the function at the origin. With this in mind we now consider the relationship of the function to the 45 degree line for different values of r .

If $0 < r \leq 1$, the slope at the origin is less than or equal to that of the 45 degree line. So, for $x > 0$ the function lies everywhere below the 45 degree line. In these uninteresting cases x_t converges on the origin irrespective of the initial value of x_0 . Thus, x is unsustainable and the population becomes extinct.

It is only when r takes a value greater than 1, that non-trivial behaviour emerges. For $1 < r \leq 2$, the 45 degree line also intersects the function at a point before or at the maximum.⁹ Now x_t converges to the fixed point $x_{s2} = (r-1)/r$. This is the case for *all* initial values of x_0 (excepting $x_0 = 0$ and $x_0 = 1$ which map

directly to the origin). Equilibrium is restored following any random disturbance, with x_t monotonically approaching it. The time path in phase space is shown in figure 3.

Figure 3: Phase space $0 < r < 2$, e.g. $r = 1.6$

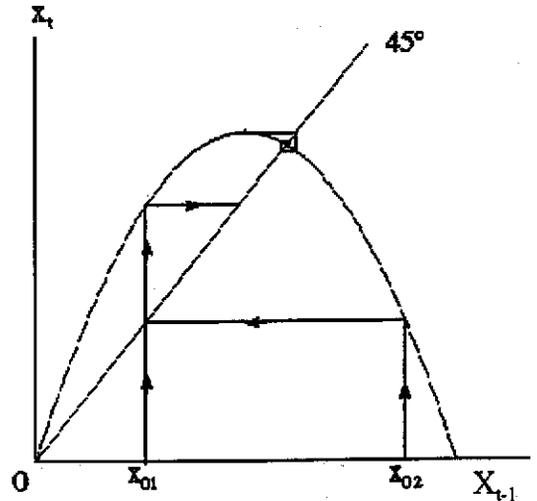


With r greater than 2 but less than 3, the 45 degree line intersects the function on its declining section. The slope of the function at this point is found by substituting (4) into the derivative of (3):

$$\frac{\partial x_t}{\partial x_{t-1}} = 2 - r \quad (5)$$

The slope at x_{e2} , for $2 < r \leq 3$ lies between -1 and 0. The absolute slope is less than that of the 45 degree line. There is a qualitative change in the behaviour of x_t , as shown in figure 4. Here, x_t initially rises towards and then oscillates to the stable fixed point in a manner qualitatively similar to damped fluctuations in the linear cobweb model. With r set exactly equal to 3, the absolute slope values of the function and the 45 degree line are equal. Convergence to the fixed point occurs, but only after a large number of iterations. This is the borderline of a further qualitative change.

Figure 4: Stable oscillations, $r = 2.5$



When r exceeds 3, the slope at the intersection is steeper than the 45 degree line, and the fixed point becomes unstable. For example, let $r = 3.17$. Now, x_t initially fluctuates, but then settles down to a regular cycle of period two, with x_t oscillating interminably between values which are independent of the initial x_0 . The attractor¹⁰ of the system is a 2-cycle, as shown in figures 5a and 5b.

Figure 5a: Two cycle, $r = 3.17$

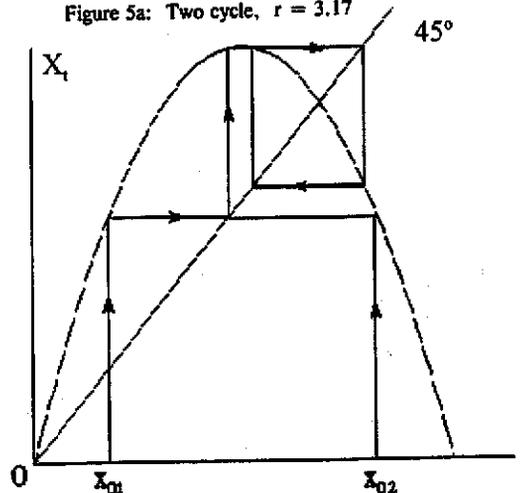


Figure 5b: Two cycle, $r = 3.17$

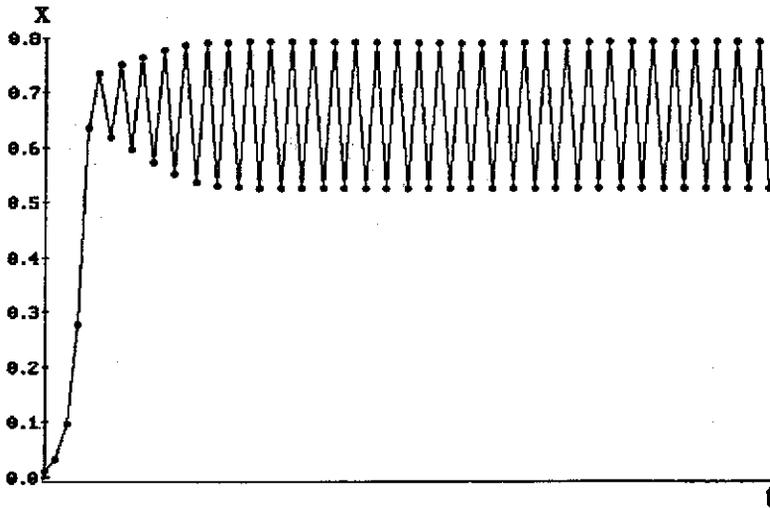
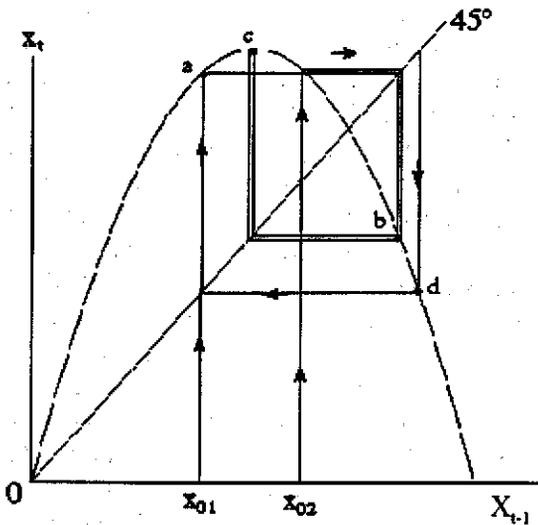
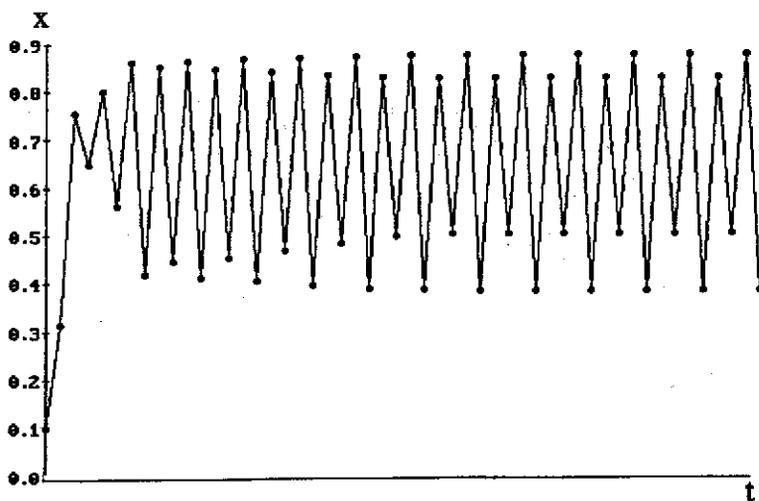


Figure 6a: Four cycle $r = 3.5$



So far, little differs from the linear cobweb model, except that explosive oscillations cannot occur, since x_t is constrained within the unit interval. In the cobweb model, behaviour is sensitive to the relative slopes of supply and demand curves, whereas in this non-linear case activity is sensitive to r . However, if r is increased further, *all similarity* with the linear model evaporates. Non-linear feedback comes into play and behaviour becomes surprisingly different. Time paths become complex, and it is not possible to infer their nature simply from examining phase diagrams, nor is it possible to derive analytical solutions. So, repeated iteration is required. In what follows, the diagrams, are to be taken as illustrating results which have been discovered and confirmed through iteration.¹¹ The inability to derive these effects by other means explains why the new found activity was for so long unsuspected. It had to await upon the development of computational power, before its rich variety could be discovered and explored.

Figure 6b: Four cycle, $r = 3.5$



The appearance of the 2-cycle is only the beginning. With further increases in r , regular cycles or attractors of increased period appear. When $r = 3.5$, there is a four period cycle as shown in figures 6a, 6b. The numerical values of this are displayed in figure 7, where it is seen that the series settles to a regular 4-cycle (at^{*}).

Figure 7: Iterated values of four cycle

Iteration	Value
39	0.382819678
40	0.826940703
41	0.500884219
42	0.874997264*
43	0.382819683
44	0.826940707
45	0.50088421
46	0.874997264
47	0.382819683
48	0.826940707
49	0.50088421
50	0.874997264
51	0.382819683
52	0.826940707
53	0.50088421
54	0.874997264

At $r = 3.56$ an eight period cycle forms. There then follow, for successively smaller increments in r , regular cycles of period 16 (figure 8)¹², 32, 64 and so on for *all* successive powers of 2. This phenomenon is defined as *period doubling*.

Finally at around a value of $r \approx 3.5699$ there occurs an infinite non-repeating limit cycle of period 2^∞ . Though this cycle is of infinite length and might be thought unpredictable, it is stable in the sense that, for alternative starting values that are *close* together, the sequence of x values generated remain also close together. For example, if we compare the time paths of x for the two alternatives, $x_{01} = 0.1$ and $x_{02} = 0.09999999$, with $r = 3.5699$, we see that the two series remain very close together even after 600 iterations of equation (3). This is seen by plotting the *differences* between the two series. After a few observations the differences between the two series become very small indeed (note the smallness of the vertical range: $-4E-05$ to $8E-05$), as shown in figure 9.

However, as r is raised above 3.5699, series

Figure 8: Sixteen cycle $r = 3.5679$

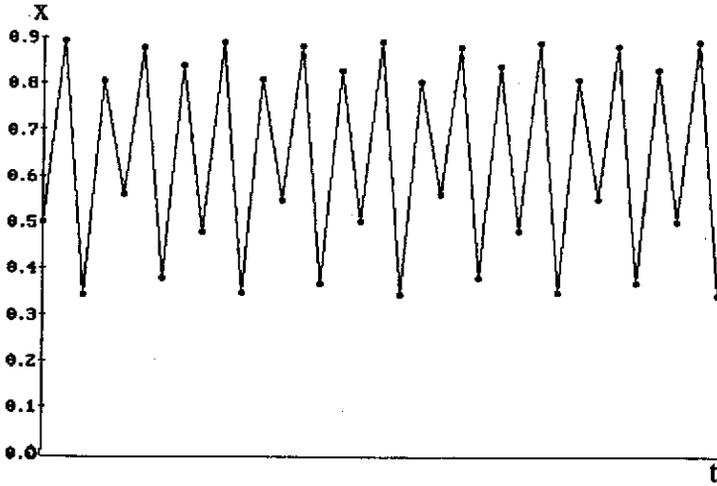


Figure 9: Difference, $X_0 = 0.1$, $X_0 = 0.999999$, $r = 3.5699$

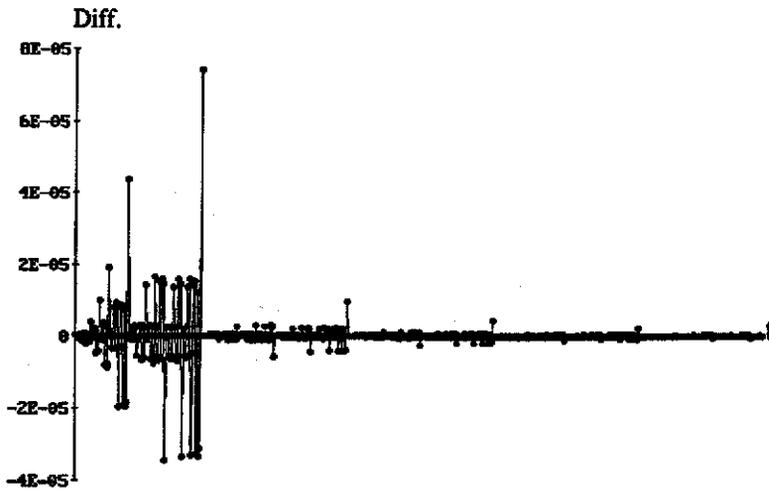


Figure 10: Sensitive dependence on X_0 , $r = 3.76$, $X_0 = 0.1$, $X_0 = 0.999999$

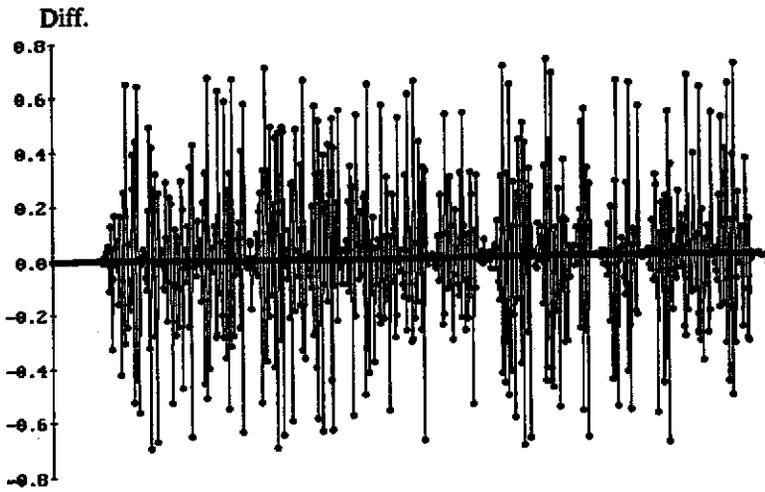
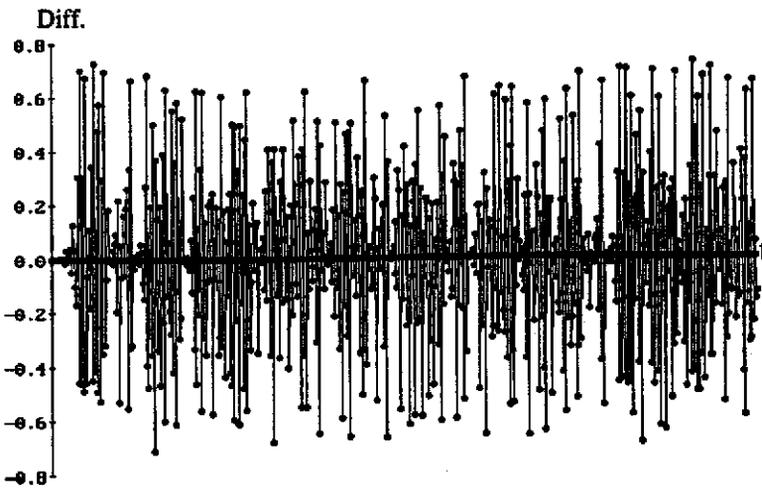


Figure 11: Sensitive dependence on r , $X_0 = 0.1$, $r = 3.76$, $r = 3.761$



degenerate. Truly *chaotic* behaviour appears. Refer back to figure 1 ($r = 3.68$, $X_0 = 0.3$), x_t displays no discernible pattern. Such attractors are known as *strange attractors*. This name was first introduced by Ruelle and Takens (1971). For our purposes their main properties are: (i) trajectories are drawn into and contained within the attractor (in this case the unit interval), and (ii) points on trajectories that are initially close do not remain close but rapidly diverge, come back close and diverge again in a continuously unpredictable manner, rather like matchsticks thrown into eddies below a waterfall. Thus, x_t is an infinite, never repeating non-cyclical series. Its time path appears random, but there is no element of randomness whatsoever. Also remarkable are the patterns. There are periods where successive values have the semblance of being regular but then degenerate into irregularity and vice versa. Not only this, but the pattern of regular/irregular behaviour is itself unpredictable. If the model represented a complete specification of a real system, in the presence of *deterministic chaos*, there would be no way to provide reliable forecasts of the real variable. In principle, this would be possible if r and a single value of x were *precisely* known. However, in real systems, these can never be known absolutely. The slightest measurement errors could lead to major failures in prediction.

At higher values of r , x_t is sensitive to the value x_0 . This is known as *sensitive dependence upon initial conditions*, or alternatively as the *butterfly effect*. This is suggested by the analogy that, given that weather systems are non-linear, the flapping of a butterfly's wing could initially generate minimal air turbulence. The effects of this might not die away, but set up repercussions which could lead to the development of unpredictable tempest. The reason lies in

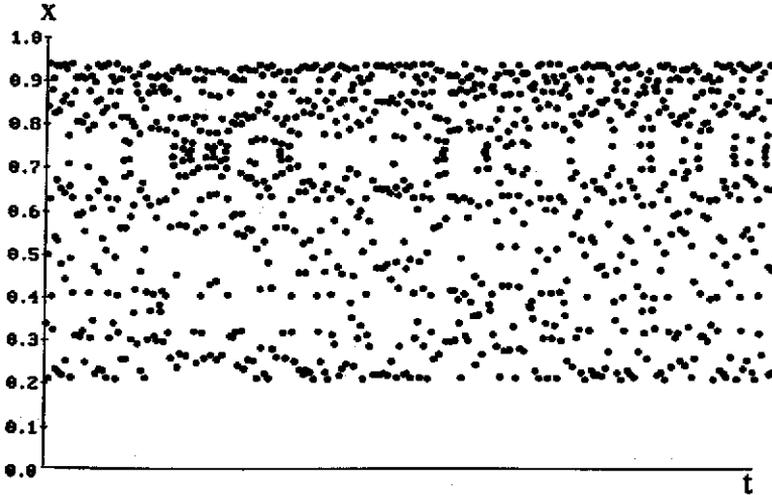
the non-linearity of the system. In the chaotic range, x_t is an infinite non-repeating series. So, no matter how close, *numerically*, two alternative values of x may be, they will lie on different trajectories and be at far removed positions within the series. Consequently, their successors in the series may differ greatly. Thus, if at a point in time a weather system is chaotic then the weather will be unpredictable.

Sensitivity to differences in x_0 is demonstrated in figure 10. This shows the difference between two segments of the series generated when $r = 3.76$, with $x_0 = 0.1$ and 0.0999999 respectively. In figure 10 the differences are plotted for 600 iterations. The divergence is, remarkably, often as much as eighty per cent, yet sometimes almost zero (note the vertical range, -0.8 to 0.8 , is much greater than that of figure 9).

Sensitivity to minimal differences in r can be seen by comparing the different series generated when $x_0 = 0.1$, with r taking the values of 3.76 and 3.761 respectively. The time paths are very different. There is minimal difference in r , but extreme divergence, (figure 11).

The absence of regularity is seen dramatically if we show the plot of a very large number of iterations (here equal to 1036), as in figure 12, where $r = 3.76$ and $x_0 = 0.1$. A Royal Academician might see something pleasing here, but to others it is surely formless and random.

Even if the laws governing the system were known and precisely formulated, the presence of non-linearities confounds prediction, since prediction requires an unattainable complete accuracy of measurement. Indeed, simulations run on different computers may differ because of differences in the ways in which different makes perform the rounding operation.

Figure 12: Chaotic series, $r = 3.76$, $X_0 = 0.1$ for 1036 iterations

4. Order within chaos

More surprises await. At $r = 3.7385$ periodicity returns with a 5-cycle. Further increases in r bring a 10-cycle, then 20, 40, 80 cycles and so on. Period doubling continues, then chaos again. Increase r further and order returns with a cycle of period 3 at $r = 3.83$. Period doubling reappears with a 6-cycle at $r = 3.845$, a 12-cycle at $r = 3.846$ and so on, until chaos reemerges. The term *chaos* is misleading since the system is completely deterministic. It would be better to say *unpredictable*. A given value of r and of x_0 completely determine all values of x_t . Behaviour is precisely replicated no matter how many times the simulation is performed on the same computer. There is no randomness whatsoever, though series often appear as if they are random. The problem is that r and x_0 can never be known with sufficient precision to make accurate predictions of real systems.

Initially, it seems unbelievable that this simple equation can generate such rich behaviour. The sources of this complexity are

clarified by perusing figures 13 and 14 where $r = 3.76$ and $x_0 = 0.2$. As we know, this is in the chaotic range and figure 13 clearly shows this. However, peruse the simplified figure 14 and trace the initial path of x_t around the diagram. It becomes clear how the curvature or non-linearity produces diversity, throwing points up high or low deterministically, but for all appearances random.

Because the patterns generated are *not* random, it might be suspected that there would be some underlying structure. It is now known that this is so, and that the appearance of cycles follows a clear and consistent pattern. Li and Yorke (1975) show that if there is a cycle of period 3, then there exist cycles of all other periods. This is a special case of a result shown independently by Sarkowskii (1964); Given the ordering of figure 15,¹³ if there exists a cycle of period q , and p precedes q in that ordering (i.e. $q > p$), then there exists a cycle of period p . Thus, the first cycles to set in are, 1, 2, 4, 8,.... It can be seen that if, for example, there is a cycle of 18, then there must also be cycles of 22,

Figure 13: Chaos, $r = 3.76$, $X_0 = 0.2$, first 100 iterations

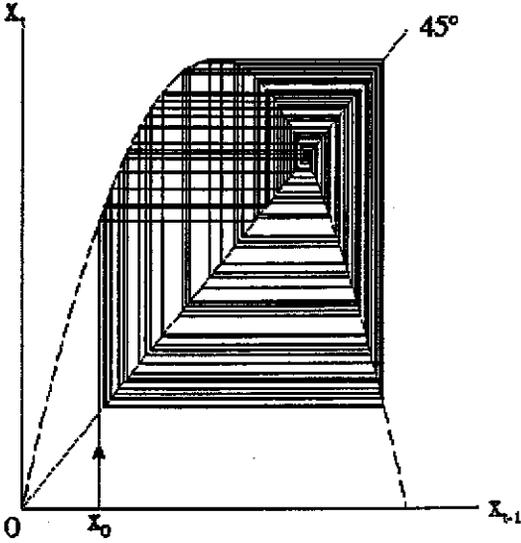


Figure 14: Chaos, $r = 3.76$, $X_0 = 0.2$, first 10 iterations

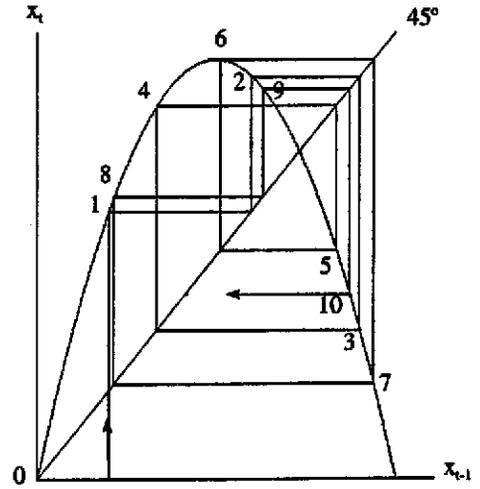


Figure 15: Sarkowskii ordering of positive integers

	3	>	5	>	7	>	9	>	11	>	13	>	>			
>	>	6	>	10	>	14	>	18	>	22	>	26	>	>		
>	>	12	>	20	>	28	>	36	>	44	>	52	>	>		
>	>	24	>	40	>	56	>	72	>	88	>	104	>	>		
>	>	48	>	80	>	112	>	144	>	176	>	208	>	>		
>	>												>	>		
>	>												>	>		
>	>												>	>		
>	>	128	>	64	>	32	>	16	>	8	>	4	>	2	>	1

26, 12, 20 and so on through all preceding values. Furthermore, it is seen from this ordering that if a cycle of period 3 exists then there exist cycles of *every other* period.

Remarkably, this ordering applies not just to the logistic equation but to all equations characterised by a single hump,¹⁴ and is an example of *universality* in chaos.

Figure 16 summarizes. The value of r is on the horizontal axis, and the associated attractors of x on the vertical. For values of $r < 3$, there is a single fixed point value of x which increases with r . At $r = 3$ the fixed point becomes unstable, the line splits and a stable two cycle appears. So, at $r = 3.17$ we can read off the two periodic values x_1 and x_2 . Further *bifurcations* or period doublings occur to give stable 4, 8, 16 cycles and so on up to the 2^∞ cycle.¹⁵ A more detailed picture is shown in the computer simulation of figure 17a.¹⁶ A vertical slice shows plotted x values for a given r . The period doubling is clearly seen, as is the degeneration into chaos when the data becomes *aperiodic* at around $r = 3.57$. Then, at around $r = 3.83$ there is a *window* in chaos as the 3-cycle emerges. Further increases in r produce period doublings giving 6, 12, 24, $3(2^n)$ cycles. As r increases further there is a reemergence of chaos, another window with a cycle of odd period, bifurcations, chaos, window, and so on.

This rich structure seems quite remarkable, but there are further properties of interest. Feigenbaum (1978) discovered that the phenomenon of period doubling occurs in accordance with the Feigenbaum constant:¹⁷

$$F = \lim_{n \rightarrow \infty} \frac{r_{n+1} - r_n}{r_{n+2} - r_{n+1}} = 4.669...$$

where r_n is the value of r at which a cycle of given period k first appears, r_{n+1} a cycle of period $2k$, r_{n+2} a cycle of $2^2.k$ and so on. Thus, we can calculate the values of r at which cycles of given periods appear. This is another example of universality within chaos, since the constant applies to *all* functions with a single hump.¹⁸

Another property is *self-similarity*. Consider

figure 17b which shows, enlarged, the area contained within the rectangle superimposed on figure 17a. Similarly figure 17c is an enlarged portion of 17b. Zooming into the figure like this shows that each smaller portion is self-similar, i.e. a copy of the whole. This property is a common feature of chaotic systems and fractals.¹⁹

Figure 16: Outline bifurcations

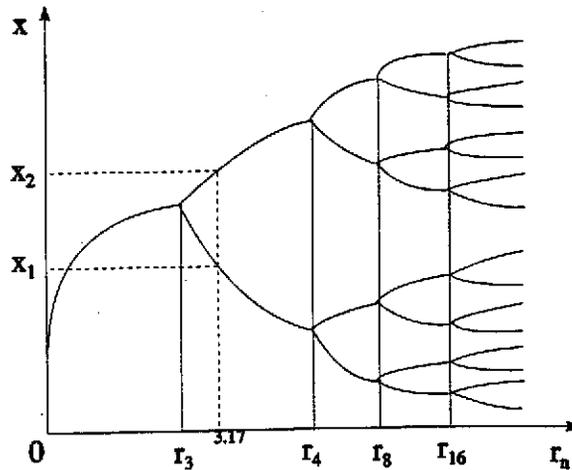


Figure 17a: Detailed bifurcations

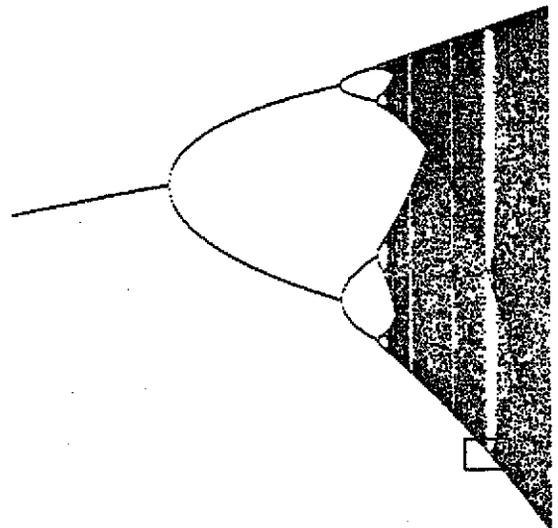


Figure 17b: Self similarity, c.f. fig. 18a

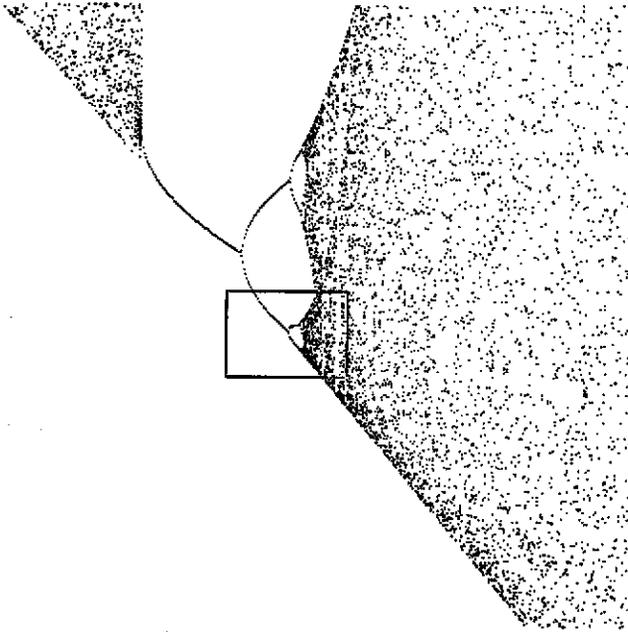
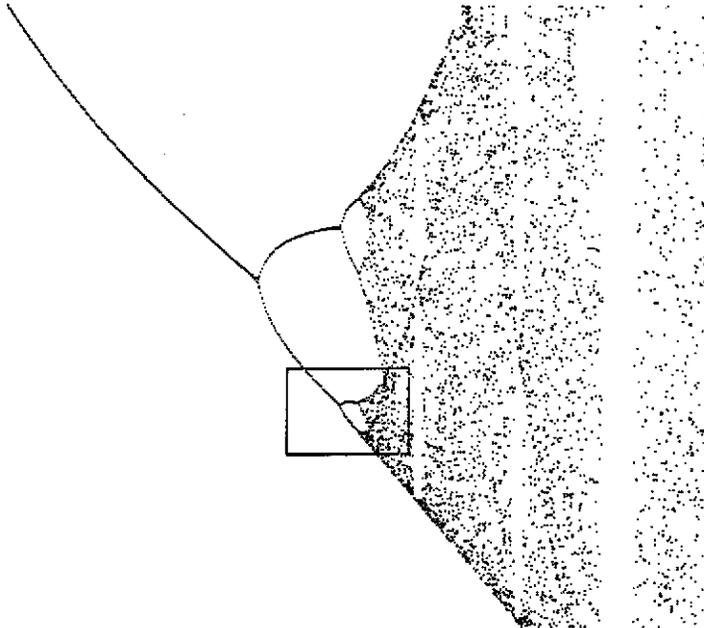


Figure 17c: Self similarity, c.f. fig. 18b



Let us draw together some of the features of non-linear systems. Simple non-linear equations can generate rich, varied patterns, much more complicated than those of linear models. Tuning the parameter r shows that series can settle to fixed points, display cycles of any period or degenerate into chaos. These patterns are never random, but completely determined by the equations and initial conditions. The behaviour of a *real* chaotic system is unpredictable because of the need to make measurements to an impossible degree of accuracy. Thus, behaviour is chaotic when it is both aperiodic and sensitively dependent upon initial conditions, though still deterministic and non-random. Non-linear models can reflect qualitatively the sort of irregularity found in natural and socio-economic phenomena better than linear models. Chaotic series display no obvious trends and are characterised by sudden change. Apparent stability may abruptly turn to give the semblance of cyclical behaviour or of randomness and vice versa. All this occurs in a quite unpredictable manner.

5. Chaotic behaviour and economic methodology

What is the relevance for economics? In some circumstances models may perform satisfactorily. For low r , patterns are not sensitive to x_0 , are of low periodicity and might be empirically corroborated. Models may in those circumstances be useful as predictors of what is fairly regular activity. However, for *some* values of r , behaviour becomes sensitive to x_0 and prediction is impossible. Predicted behaviour is so sensitive to imprecisely measured values of x_0 that it is unlikely to bear correspondence with reality. Attempts to isolate stochastic influences are fruitless since there is no clear trend to be exposed.

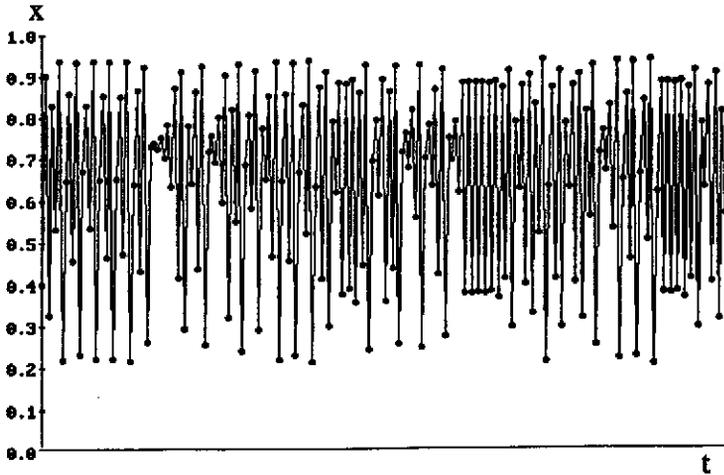
Alternatively, it may be impossible to

specify models accurately enough, even if precise measurement were possible. If estimates of r are imprecise, or if there are factors causing r or M themselves to vary, then prediction will fail even if X were accurately known. If in (3a) X were fish it is conceivable that they could be counted without error, but how could it be possible to estimate M without error? So, for any error in the estimate of M there is an implied error in $x = X/M$ in (3b). Such errors may be impossible to avoid, yet we have seen that minimal differences in x_0 can lead to widespread discrepancies in prediction.

Models that can correctly explain may, nevertheless, lack predictive content. It is possible that a model which performs well at some times fails at others. One concludes that if it is impossible to get predictable output from such a simple idealised system as this, then there is an obvious greater difficulty in the modelling of *real* systems where non-linearity is present.

If non-linearity is important then modelling economies by making linear approximations to make the mathematics tractable is doomed to fail. Systems which appear ordered may suddenly and inexplicably deteriorate into chaos. If irregularity is observed, there may be no clear underlying trend to be extracted. For example, in fluid dynamics the Navier Stokes partial differential equation approximates fluid flow, and predicts behaviour consistent with experiment, but only up to a critical flow, thereafter turbulence breaks out and the model fails.²⁰ In economics the clearest example is seen in the paucity of the *predictive* content of the linear cobweb model and derivatives of it.

In science, experiments can be precisely replicated, but this is impossible in social sciences and represents a major difference between their methodologies.²¹ Inferences have to be drawn from historical rather than

Figure 18: Chaotic series, $r = 3.765$, $X_0 = 0.4$ 

experimental data. This leads to problems of a different nature.

Consider figure 18. This series was generated with the values $r = 3.765$ and $x_0 = 0.4$. Looking at the plot, it can be seen that, within it, there are sections of apparent regularity. There are instances where there is a clear semblance of a 2-cycle, but as we know this is an illusion: the output as a whole is in the chaotic regime. Suppose this plot represented the behaviour of some real phenomenon, and that data had been collected for only the part resembling the 2-cycle. It might then be an easy matter for some researcher to construct a theoretical model to explain this apparent cycle, and predictive success might guarantee that researcher's acclaim. However, as time proceeds, further observations seem at odds with the theory. What was once a 'good' theory falls into disrepute, and is either modified by invoking *ad-hoc* explanations for the collapse of the trend, or discarded as having been falsified,

and there is a search for a new theory. However, as we know, for this data set the truth is that there never was any explanation. The data is determinate but unpredictable and a theory that 'explains' a part of it cannot explain the whole. Economics is replete with theories that have appeared to work well in retrospective explanation, but fail miserably in the predictive stakes. The breakdown of the Phillips curve is an obvious example. Now this is very worrying, because at a philosophical level it reinforces the fact that induction can never be relied upon as a provider of ultimate proof. Observations represent, always, only a sample of a greater whole. If regularity is observed this does not guarantee a continuance of that regularity, and current explanations may later prove to be incomplete and/or incorrect. Planetary orbits display well understood regularities, but what faith can be had in their continuance? At a general level, it is taken as axiomatic that they are immutable, and so there is an inbuilt

assumption of permanence which conditions the ways in which we organise our lives. We do not expect the next ice age tomorrow! However, if this behaviour were to represent only a phase of regularity within a greater time scale, then explanations currently successful in predicting may nevertheless be incorrect.²² Thus, if we observe activity that can be explained successfully by linear methods, we can never guarantee that the system *is* linear. If on the other hand behaviour is non-linear, then a predictive model may be a chimera.

6. Chaos and research in economics

It is known that many natural phenomena are governed by non-linearity. So, can economic activity be explainable in the same terms? Much economic activity is highly complex with underlying relationships complicated by stochastic factors. *A priori*, there would seem no reason to suppose that economic systems are any less governed by non-linearity than non-economic ones. If economic systems formerly thought to be complicated or random are indeed non-linear and deterministic, then the question arises whether it is possible to unearth that non-linearity. This cannot be easy. Standard tests of randomness often fail to discriminate between chaotic and random series. Simple equations can generate complicated series. It is less easy to find simple(?) relationships that model observed but complicated data. Discovering any underlying non-linear equations is difficult, particularly if real data also embrace elements of randomness. Few would claim that everything in life is completely predetermined. Even if it were, we could not *know* it, and could have no way of constructing a theory to endogenise those deterministic events which, to us, must *seem* random. It has been usual to *assume* that real systems are subject to unanticipated stochastic

influences. If, however, there is also sensitive dependence, the detection of any underlying equations is likely to be even more tortuous, for any random effect will disturb those conditions. This is a greater problem in economics, since experiments cannot be replicated and there is an absence of controlled conditions. The chaotic motion of a turbulent fluid is less likely to be subject to unknowable random influences than are financial variables. Thus, the death of a prime minister, the collapse of a government, a ministerial statement or indiscretion etc. are well known to have an exogenous effect upon economic data. If non-linearity is important any such event may cause widespread and unforeseeable repercussions.

Against this background, research is undertaken. There are two main lines of interest. One involves constructing theoretical models to show how chaotic dynamics could arise within economic systems. The other involves the development of procedures to detect non-linearity in empirical time series. It is beyond the scope of this essay to discuss this burgeoning, detailed, technical and still largely inconclusive literature.²³ However, it is possible to provide some pointers to the directions of research.

The literature on non-linear theoretic modelling is eclectic. The prime purpose has been to demonstrate how reformulating orthodox models can generate chaos. Since the essential message is that simple systems can generate complicated dynamics, many of these models are deliberately simple and based upon the logistic. A model of an economic system is built, plausible and/or non-contentious assumptions of non-linearity are introduced, a transformation of variables is made and the model recast in the form of the logistic. The ability of these models to produce chaos provokes the suggestion that observed diversity in real economic time

series might be deterministic and endogenous rather than random or stochastic.

There are many papers, and we mention only a few. Savit (1988, 1989) used the logistic equation as a model of commodity prices, where x_{t-1} represents price in period $t-1$ and $r > 1$ a growth parameter. The simple economic rationale is that sellers attempt to push price up to $r \cdot x_{t-1}$ in period t , which is resisted by buyers to the order of $-r \cdot x_{t-1}^2$. Baumol and Quandt (1985) propose a model of advertising expenditures, where profit is a quadratic function of advertising, with advertising a fixed proportion of profit in the previous period. Jensen and Urban (1984) reformulate the linear cobweb model by introducing non-linear supply or demand curves to yield the logistic equation. Clearly these models can generate time profiles which are not qualitatively dissimilar to empirically observed series. Currie and Kubin (1995) use the logistic equation to show that where two markets are independent in supply and apparently negligibly interdependent in demand, chaotic behaviour could arise. This suggests that, even where markets are considered to be practically independent, taking a partial approach to one of the markets may not enable its qualitative time path to be discovered. Benhabib and Day (1981) develop a model which shows that if there is some cyclicity in preferences, erratically fluctuating choices can arise which are deterministic and chaotic rather than random or exogenous. Tofallis (1995), in a model simpler than the logistic, shows that chaos could arise in a bank account where interest accrues at a fixed rate, subject to a fixed withdrawal once the account exceeds some threshold.

All in all, theorists have demonstrated that it is possible to think of real economic phenomena for which non-linear equations

could provide a first approximation. Thus, chaos could be claimed not to be an aberration but to be a distinct possibility within economic systems, and so empiricists have prospected for it.

In attempting to detect chaos in economic series there are a number of lines of attack. Consider figure 19a. This contains numerical values from the chaotic series of figure 1. Both the numerical values and the picture *seem* random. No pattern is readily apparent in either. In figure 19b each numerical value is plotted on the x axis, against that of its successor on the y axis. It is seen that plotted points lie precisely upon the curve $y = 3.6x - x^2$. This is unsurprising, because this is precisely the relationship determining the successive values. If instead, this data had been empirically gathered and represented some economic variable, then the uncovering of a relationship such as that of figure 19b would be strong evidence for the presence of non-linearity. If on the other hand the data had been truly random, no relationship could

Figure 19a: Deterministic data

$$x_t = x_{t-1} (1 - x_{t-1}) \quad r = 3.6 \quad X_0 = 0.3$$

Iterations 50 to 65

0.6206991
 0.866388756
 0.425994148
 0.899845132
 0.331655842
 0.815709699
 0.553204779
 0.909582845
 0.302650165
 0.776675197
 0.638299073
 0.849613988
 0.470193818
 0.916730657
 0.28091486
 0.743366261

Figure 19b: Lagged plot of deterministic data of fig. 19a

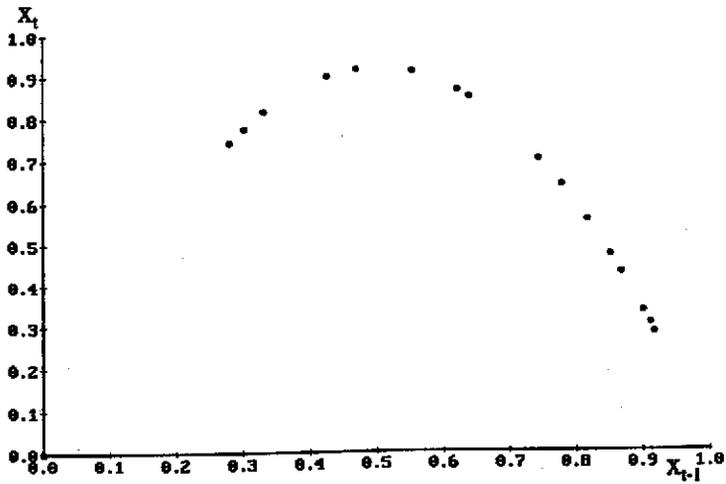
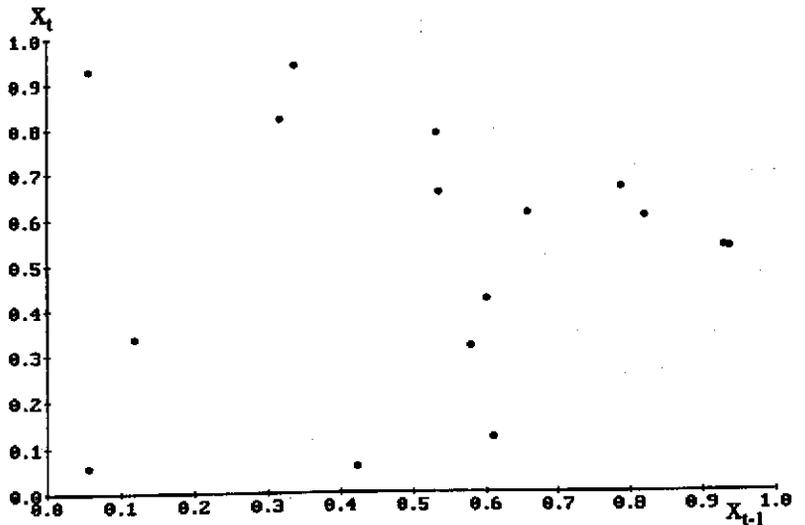


Figure 20a: Generated random data

```

0.378084107
0.375344519
0.71451913
0.503704915
0.097296041
0.755746857
0.401212333
0.517651567
0.067364131
0.794460914
0.611960683
0.277191382
0.507893005
0.228525304
0.558332975
0.599470623
    
```

Figure 20b: Lagged plot of random data of fig. 20a



be uncovered. The data of figure 20a have been randomly generated, and on the face of it are not readily distinguishable from those of figure 19a. However when the values are plotted in figure 20b, there is nothing to suggest any causal inference.

It would be facile to pretend that the detection of non-linearity is as simple as this. There are many problems. There is good reason to suspect many economic variables are in more complicated relation. For example, Peters (1991) discusses the logistic delay equation which involves a two period lag: $X_t = aX_{t-1}(1-X_{t-2})$. Even if the lagged structure were known, because of stochastic influences and measurement errors, empirical data would never be found to lie precisely upon a given curve. Thus the exact formulation of any curve can never be discerned. A sine curve or a quadratic may equally approximate the data, and unless there are good theoretical reasons for preferring one over the other it could be anybody's guess. Indeed, it is conceivable that the best fit curve is one that is theoretically less sustainable. Yet, the slightest variation in the structure and/or parameters of a model can produce widely variant predictions. Just because data is chaotic does not imply an absence of structure, just the reverse. What is of great difficulty is uncovering that structure.

Another approach is to look for evidence of period doubling. This has been observed in a number of physical systems, and provides good evidence of non-linearity. If this were found in economic data then it would likewise, provide good evidence. There are problems. Data sets are seldom as large or as complete as available in the sciences. Recall also that, for equations with a single hump, period doubling appears at a rate determined by the Feigenbaum constant. Recall also that this is universal and applies to all equations

with a single hump. Thus, observed period doubling is consistent with a whole class of curves, and though period doubling would provide strong support for the presence of non-linearity, it would provide no indication of the particular generating equations, even though a statistical curve could be approximated.

Another avenue involves trying to discover the extent to which trajectories diverge given slight variations in initial conditions. Weather systems have been shown chaotic. The gathering of meteorological data is imperfect, and with sensitive dependence upon initial conditions the weather becomes unpredictable beyond a few days. This is one of the reasons for the discontinuance of *routine* long-range forecasts. At other times weather patterns can be stable for lengthy periods. A procedure helpful in forecasting, is to run simulations a number of times, each with slight variations in initial conditions. If these produce fairly consistent predictions, then the weather can be said to be within a stable pattern and a forecast made. If widely divergent predictions arise, the weather is within a chaotic phase and unpredictable. If those times when chaos is existent can be known, this is helpful not in prediction, but in *knowing* that in the current regime prediction is impossible.

Researchers have attempted to detect non-linearity in economic systems. Most of this research has concentrated upon macroeconomic and financial time series. For example, Brock and Sayers (1988) are concerned with the detection of non-linearity in the business cycle and Scheinkman and LeBaron (1989) with stock returns. In a more recent appraisal of research LeBaron (1994 p.397) notes that as yet, 'Studies have found little or no evidence for chaos in any economic time series, but they have turned up a surprising amount of unexplained nonlinear

structure in many series'. He points out some of the difficulties. In macroeconomics the ability to estimate non-linear processes is impeded, since noisy time series arise from systems where the dynamics and the methods of measurement are subject to change. In financial markets traders have some ability to perceive complex patterns which they then trade against. This reduces and obscures further any underlying patterns and makes forecasting even more difficult. Despite these negative findings, he argues that it remains important to be able to estimate what fraction of fluctuations arise from exogenous shocks as opposed to the underlying structure of a system. He concludes that research should now be directed towards the examination of multivariate data sets, to develop closer links between theory and empiricism, to develop further statistical tools and to explore the dynamics of large scale interconnected systems. Brock (1990 p.446) concluded that:

The 'problem' for the argument, that the randomness we see in economic time series is due to low-dimensional deterministic chaos, is the presence of intelligent, farsighted humans. Their desire to smooth consumption and production over time creates negative feedback loops. Such intertemporal arbitrage operations tend to smooth out deterministic fluctuations. This is the basis for the belief of many economists that deterministic fluctuations cannot be an important component of any economically sensible explanation of the fluctuations that we see in economic time series, especially asset returns.... While at an anecdotal level it seems obvious that chaos and instability are common in interactive social behaviour systems such as economies, it is much harder to document the presence of chaos and instability in economic data. To my

knowledge, although claims have been made, this has not been done in a scientifically convincing way.

Theoretical modellers seldom claim their models represent a particular reality, for the main import of theory has been to show how, in principle, chaos could arise. However, Brock (1990 p.428) states that while:

...theoretical models can be constructed that generate competitive equilibria that follow these difference equations, they do not demonstrate that the values of model parameters needed to generate chaotic equilibria can be found that are consistent with empirical studies. To put it another way, no one has estimated a class of models that allow chaotic behaviour and found estimated model parameters consistent with chaotic dynamics. This lack of empirical 'cross validation' is a serious problem.

Thus, for the moment, research only cautions against predictions of orthodox economics, rather than show them false.

7. Conclusions

Chaos theory has brought understanding and problems for analysts in many fields. To make real systems tractable, the method of making linear approximations has provided successful predictors for many situations, but in other situations it is untenable. This is where small variations or errors in observed variables result in large discrepancies in predicted variables. There is a dilemma. In these circumstances linear models become bereft of predictive content, since they imply a type of regularity that is non-existent. If non-linearity is acknowledged it becomes impossible to construct models capable of predicting under all circumstances. Indeed there are instances where, even if the *true* equations were known, prediction would remain impossible due to unresolvable

inadequacies in measurement. Whilst much of natural science has proved tractable to linear modelling, economic phenomena have not. Chaos has shed light on previously intractable natural phenomena (e.g. turbulence and weather systems), but the same cannot, yet, be said for economics. The discovery of chaos and the study of non-linear systems have changed the ways in which natural scientists view the world, it remains to be seen to what extent this will be the case in economics.

Endnotes

1. Manchester Metropolitan University. I am particularly grateful to John Avery, and to Susan Evans, Derek Leslie and George Zis, also to the editor and referees. I am responsible for all errors and deficiencies.
2. Kelsey (1988) offers a concise technical treatment for economists. At the popular scientific level Gleick (1988) is an acclaimed account of the issues and personalities, whilst Stewart (1990) is the best introductory survey. The most comprehensive treatise that does not make excessive demands on non-mathematicians is Peitgen et. al. (1992), who also include simple Basic listings. Hall ed. (1991) has a varied selection of articles from *New Scientist*.
3. For methods of solution see Baumol (1970) or Sandefur (1990).
4. This is no idle jest. Leading architects of chaos theory made clandestine attempts to outsmart roulette wheels in Las Vegas. For a fascinating account of this adventure see Bass (1991).
5. Extended discussion of the philosophical issues is in Kellert (1993)
6. Sometimes known as the Verhulst equation, after the nineteenth century Belgian mathematician P F Verhulst, or as the May equation after the Oxford biologist Robert May who used it in the study of population dynamics.
7. For clarity, the horizontal and vertical scales differ in fig. 3 and others. Thus, the 45 degree line *appears* steeper than 45 degrees.
8. The print outs (e.g. fig. 1) were generated from the simple spreadsheet within *Microsoft Works*. Many aspects of chaos can be explored with Basic, and without the need of sophisticated programming skills. Just a few lines of Basic produce fig. 6a. The package *Fractint* (Tyler, et. al. 1992), freely available within the public domain, enables many aspects of chaos and fractals to be explored via the desktop, as does Wegner and Peterson (1991) which includes a disc of, and a guide to, the *Fractint* package.
9. From (4) this intersection occurs at $x_{e2} = (r - 1)/r$. Given $0 < r \leq 2$ then $x_e \leq 1/2$. Since the maximum is at $x_{r-1} = 1/2$, the intersection occurs at or before it.
10. The term 'attractor' is used to define the set of points which constitute the equilibrium of a dynamical system. In the case of a point attractor the system converges to a single point as in figs. 3 and 4. When a system settles to form a repeating cycle, as in fig. 5, the attractor is said to be a limit cycle.
11. The computer used was a Mitac desktop with a 386sx 25Mhz processor.

12. In fig. 8 transients are omitted. The values of x are those after sufficient iterations to discern the cycles.
13. The ordering is such: The first row is the odd integers except 1. The second is their doubles. The third is given by multiplying them by 2^2 , so doubling them again. This is continued for all powers of 2, each row doubling the previous row. The final row is given by all the powers of 2 in descending order to unity.
14. E.g. the ordering equally applies to the trigonometric equation: $x_i = k.\sin(x_{i-1})$.
15. For a more detailed but accessible explanation of how a cycle of given period becomes unstable at a given value of r see May (1976).
16. The patterns of figs. 17a,b,c were generated using the package Fractint (Tyler, Wegner *et. al.* 1992)
17. Peitgen *et. al.* (1992, pp. 610-618) discuss the nature, calculation and universality of the Feigenbaum constant.
18. E.g. the Feigenbaum constant applies equally to period doubling in the iteration of: $x_i = k.\sin(x_{i-1})$.
19. The term 'fractal' is used to refer to shapes which are irregular, but where the shape of the parts somehow reflects the shape of the whole, and therefore displays self-similarity. Examples are the branches of a tree, parts of a cloud or stretches of a coastline, all of which if magnified would display shapes which, though not identical, are similar to the whole.
20. For a readable and first hand account of investigations into the nature of turbulence and its relationship to chaos see Ruelle (1991). See also Mullin (1991).
21. The division of the natural and social sciences on this basis is a blurred one. Experimentation is not possible in geology and palaeontology which are retrodictive. Psychology frequently invokes experiment. There are attempts to develop an experimental economics (Loomes 1989). Despite this, experiments in social science are less likely to be capable of precise replication than experiments in the 'hard' sciences, and so the division is a useful one.
22. There is indeed evidence to suggest that the behaviour of the solar system is not as predictable as formerly believed and that planetary orbits are subject to chaotic dynamics (see Murray 1991).
23. See Benhabib (1992), for a collection of papers demonstrating the varied directions and breadth of research. Bullard and Butler (1993) survey macroeconomic models. Byers and Peel (1994) survey applications to industrial economics.

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