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Monotonic Movement of Price Vectors

Christian Bidard and Ian Steedman[1]

Abstract
It is well known that long-period relative prices in a Sraffa-like analysis can change in an apparently complicated way as the uniform rate of profit (interest) changes. It is shown here how a certain simplicity of movement (monotonicity) can be detected.

1. Introduction

As the uniform rate of profit changes in almost any Sraffa-like system, the relative long-period prices of commodities have to change as well; this fact has of course been central to certain criticisms of some well-known marginalist theory for, at root, reswitching, capital-reversing and so on are but varied manifestations of this dependence of relative prices on distribution. Yet to say that relative prices depend on distribution is not necessarily to say that little or nothing can be asserted about the nature of that dependence. But this stronger claim has been made; Pasinetti (1977, 82), for example, has written that the question how relative prices change with the rate of profit 'is not a question to which any simple answer can be given' and Mainwaring (1984, 79) has written similarly that 'the pattern of price variation is a complex one and, except in very special cases, nothing *a priori* can be said about it'. It is shown below that this stronger claim is *too* strong, because one can in fact detect a significant element of simplicity in the relative price/rate of profit relationship.

Consider a square production system in which commodity prices and the rate of profit are uniform throughout the economy. Let the relative wages of the different kinds of labour be fixed, so that one may argue as if labour were homogeneous. The wage/price/profit rate relations for the system may be written as:

$$pB = w\,e + (1+r)pA \qquad (1)$$

where p is the row vector of prices and e that of employment levels process by process. The columns of B and of A show the outputs from and produced inputs used up in the various processes; w is 'the' wage rate and r is the uniform rate of profit. Given only that (B-A) is non-singular, (1) may be rewritten as:

$$p = wl + rpH \qquad (2)$$

where $l \equiv e\,(B-A)^{-1}$ and $H \equiv A(B-A)^{-1}$. In any viable single product system H is semi-positive. Interestingly, however, such semi-positivity of H plays no role in the following argument, which can allow for joint production and for negative elements in H.[1] We shall assume, though, that H has a positive real characteristic root, h_1, which is greater in modulus than any other real root and than the real part of any complex root. The maximum rate of profit, corresponding to $w = 0$, is defined by $h_1\, r = 1$.

The *relative* prices in (2) are, of course, unaffected by changes in the nominal wage rate w but they most certainly vary with the profit rate r-and hence with any measure of the real wage rate - other than under the fluke condition that l and lH are proportional. Setting aside that uninteresting fluke case, one knows that the pattern of the changes in relative prices, as r varies, can appear to be complex. The object of this short paper is to show, nevertheless, that there is a certain simplicity in that pattern of price changes if one views matters in the appropriate way. More specifically, we are to study the conditions under which the Euclidean angle between the generic price vector and the particular price vector obtaining when $w = 0$ will *decrease monotonically* as the rate of profit increases. Rather than aiming for maximal generality, we shall seek to present this result in a very simple way, emphasizing the case of a three-commodity economy.[2]

2. A normal H matrix

Throughout this section we suppose the matrix $H \equiv A(B-A)^{-1}$ to be a *normal* matrix. By definition, this means that $HH' = H'H$, where H' is the transpose of H. It will be clear at once that if H is symmetric ($H = H'$) then it will be normal. But it is not true that a normal matrix must be symmetric. H will be normal if and only if it can be written as:

$$H = Q\Lambda Q' \qquad (3)$$

where Q is an orthogonal matrix and Λ is either strictly diagonal *or* contains one or more 2 x 2 'blocks' of a certain form along the diagonal. (See, for example, Horn and Johnson, 1990, p.105.)[3] If all the characteristic roots of H are real then Λ is strictly diagonal and, from (3), H is indeed symmetric. But if H has at least one pair of complex roots then (3) shows it to be normal but not symmetric.

Using (3), we may rewrite (2) as:

$$pQ = wlQ + rpQ\Lambda$$

or

$$\pi = w\lambda + r\pi\Lambda \qquad (4)$$

say. Since $\pi \equiv pQ$ and $\lambda \equiv lQ$ represent *orthogonal* rotations of p and l, it will be clear that the Euclidean angle between any pair of p vectors will be exactly equal to that between the corresponding π vectors; so we shall work with (4). Specifically, we suppose that there are just three commodities and that (4) may be expanded to:

$$\pi = w\lambda + r\pi \begin{bmatrix} h_1 & 0 & 0 \\ 0 & h_2 & h_4 \\ 0 & -h_4 & h_3 \end{bmatrix} \qquad (5)$$

where h_1 is positive and is the root of greatest modulus. If H has only real roots, $h_4 = 0$; if H has a pair of complex roots, $h_2 = h_3$: thus it is always true that $(h_2-h_3)h_4 = 0$.

When wages are zero, the economically interesting solution to (5) is $h_1 r = 1$ and $\pi = (1,0,0)$. Let $\theta(r)$ denote the Euclidean angle between a generic price vector π, defined for profit rate r, and the particular price vector (1,0,0). From the usual formula:

$$\cos^2\theta(r) = \left(\frac{\pi_1^2}{\pi_1^2+\pi_2^2+\pi_3^2}\right)$$

or

$$\tan^2\theta(r) = \left(\frac{\pi_2^2+\pi_3^2}{\pi_1^2}\right) \qquad (6)$$

Now (5) is readily solved for π_1, π_2 and π_3 (with $w \neq 0$) and one finds that:

(a) if H is symmetric ($h_4 = 0$) then

$$\tan^2\theta(r) = \left(\frac{\lambda_2}{\lambda_1}\right)^2 \left(\frac{1-h_1 r}{1-h_2 r}\right)^2 + \left(\frac{\lambda_3}{\lambda_1}\right)^2 \left(\frac{1-h_1 r}{1-h_3 r}\right)^2 \qquad (6a)$$

(b) if H is not symmetric ($h_4 \neq 0$) then

$$\tan^2\theta(r) = \left(\frac{\lambda_2^2+\lambda_3^2}{\lambda_1^2}\right) \left[\frac{(1-h_1 r)^2}{(1-h_2 r)^2 + h_4^2 r^2}\right] \qquad (6b)$$

But it is easily shown that, whenever $0 \leq h_1 r < 1$, the right-hand-sides of both (6a) and (6b) are monotonically decreasing as r increases; at $h_1 r = 1$ each right-hand-side is, of course, zero. Hence, whether the normal matrix H is symmetric or not, the Euclidean angle between the generic price vector $p(r)$ and the particular price vector $p(w = 0)$ decreases monotonically as the rate of profit rises from zero to its maximum value of h_1^{-1}. Geometrically, then, if the vector p $(w = 0)$ is thought of as the axis of any number of alternative Euclidean cones of circular cross section, as r rises the actual price vector always moves into a 'sharper' cone of smaller cross section, this movement being strictly 'one way'. In this clear and interesting sense, then, one can say that (with three commodities and a normal H matrix) the movement of relative prices, as the rate of profit changes, has a far more definite pattern to it than is sometimes thought.

3. Generalization

If the above result is found to be of some interest, it will at once be asked how dependent it is on our special assumptions. First, let us maintain the assumption that H is a normal matrix, with a positive root h_1 which is the root of greatest modulus, but allow an arbitrary number of commodities. Equation (4) still holds and (5) can at once be extended in the appropriate fashion. It is then easily seen that the expression for $\tan^2 \theta(r)$ will just be the sum of various terms, some being of the form shown in (6a) and some of the form seen in (6b). For $0 \leq h_1 r < 1$, each and every term in the sum will fall monotonically as r increases, reaching zero at $h_1 r = 1$. Hence our assumption of only three commodities was quite unnecessary other than for ease of exposition. It can also be shown - although the demonstration is too tedious to be included here - that the monotonicity result does not in fact depend on one of the two price vectors being that at $w = 0$ (i.e., $h_1 r = 1$). The Euclidean angle between price vectors at *any* two rates of profit, $0 \leq r_1 < r_2 < h_1^{-1}$, is monotonically decreasing in r_1 and increasing in r_2.

More pressing is the question whether one can move away from the strong assumption that H is a normal matrix. After all, no economic considerations can be offered to support that assumption. The answer to be considered here[4] is that a simple transformation can almost always be applied to (2) to produce a system which *does* involve a normal matrix. Suppose then that (3) must be replaced by:

$$H = X \Lambda X^{-1} \tag{3a}$$

where H is now only supposed to be diagonalizable; rewrite (2) as:

$$pX = wlX + rpX\Lambda \tag{2a}$$

Now let X = (UV') be the singular-value-decomposition of X where, by definition, U and V are both orthogonal matrices and, since X is non-singular, each diagonal element of   is strictly positive. (See for example, Horn and Johnson, 1990, pp411-20.)[5] We now generalize (4) to:

$$(pU\hat{x}) = w(lU\hat{x}) + r(pU\hat{x})[V'\Lambda V] \tag{4a}$$

In (4a), the matrix [V'ΛV] is normal. The transformed vectors (pU) and (lU) are obtained from p and l, respectively, by first applying an orthogonal rotation, to obtain (pU) and (lU), and then changing measurement units along the new axes, to obtain (pU) and (lU). It then follows immediately that the Euclidean angle between the generic ($p(r)$U) and the particular transformed price vector ($p(w$=0)U) will decrease monotonically to zero as r rises from zero to h_1^{-1}. It is important to note that the transformation from p to (pU) is not only simple in its meaning but is a transformation deriving from the matrix H itself (and hence, more fundamentally, from B and A in (1)); it is not a transformation imposed 'from outside' by the arbitrary choice of the theorist.

4. Conclusion

We have considered square price systems of the kind stated in (1); joint production and fixed capital are allowed, provided only that the matrix $H \equiv A(B-A)^{-1}$ has a positive root which is the root of greatest modulus and which defines the maximum rate of profit, $h_1\ r = 1$. Attention has been centred on the Euclidean angle between the generic price vector, $p(r)$, and the particular price vector $p(w = 0)$. It has been shown that, if H should happen to be a normal matrix, then it follows immediately that that angle will decrease monotonically to zero as the rate of profit rises from $r = 0$ to $r = h_1^{-1}$. If, as is far more likely, H is not a normal matrix, there is a simple transformation of the price vectors, springing from the technical conditions of production themselves, such that the transformed price vectors exhibit the monotonic behaviour just described. The variation of long-period relative prices with the rate of profit is thus more regular as has sometimes been thought.

Endnotes

1. See the various essays in Pasinetti, ed., 1980, for discussion of both the complexities of joint production and the significance of the 'vertically integrated' matrix H used in (2) and throughout this paper.

2. For a more general approach to the same issue, see Bidard and Steedman (unpublished); that more general approach is, however, far more abstract, involving the use of non-Euclidean angles, and may not appeal to most economists!

3. The 'blocks' in question are of the form:

$$\begin{bmatrix} h\cos\alpha & h\sin\alpha \\ -h\sin\alpha & h\cos\alpha \end{bmatrix}$$

for some α and some h such that $0 < h < h^1$. The complex numbers $h \exp(\pm i\alpha)$ are conjugate characteristic roots of H.

4. See, again, Bidard and Steedman (unpublished) for a more sophisticated answer.

5. Of course, one could simply say of (2a) that Λ is a normal matrix and hence that our previous analysis holds good for (pX). That is true, of course, but we proceed as in the text to bring out the role of orthogonal rotation. The analysis of section II is just the case $\hat{\Lambda} = I$ and $Q \equiv UV^t$.

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