

A Dynamical Model of Business-Cycle Asymmetries: Extending Goodwin

David Harvie, Mark A. Kelmanson and David G. Knapp¹

ABSTRACT

A rarely noted, economically unrealistic feature of Goodwin's (1967; 1972) celebrated 'growth-cycle' model is that its state variables — the wage share of output and the employment proportion — can exceed unity. We propose a novel extension of the two-variable dynamical system which ensures that its solutions remain within the economically feasible region, i.e. the unit square of the wage-share-employment-proportion phase plane. In a further extension, we obtain a model which, besides possessing a richer economic interpretation than the original, is able to generate asymmetric solution cycles. We use numerical techniques to investigate the new model's properties; in particular, we examine business-cycle deepness and steepness.

1. INTRODUCTION

The Goodwin growth cycle (Goodwin, 1967, 1972) is a macroeconomic model, analogous to the Lotka-Volterra predator-prey model, which describes population dynamics of competing species (Lotka, 1956; Volterra, 1931a,b, 1937). In the Goodwin model, the dynamic interaction is between the distribution of income and the proportion of the workforce in employment. In this model, the wage share of national income is the predator variable and the employment proportion is the prey.

Goodwin himself described his model as 'starkly schematized and hence quite unrealistic' (Goodwin, 1967, p.54; 1972, p.442). Nonetheless, despite its limitations, the model has proved to be remarkably popular and has inspired a huge literature. Part of the model's enduring appeal lies in its simplicity; in particular, the elegance with which it illustrates the cyclical relationship between income distribution and employment in a dynamic capitalist economy over the course of the business cycle. Now, four decades on, the model still inspires new contributions to the economics literature, particularly that part of it concerned with dynamic modelling (e.g., Manfredi and Fanti (2004)).

The state variables in Goodwin's model are expressed in the form of ratios: wage *share* of national income and employment *proportion*. Yet, internal to the original model, there are no forces which prevent either of these variables from exceeding unity; an outcome which is, by definition, unrealistic. In real economies, neither the wage share of national income nor the employment proportion can exceed unity. Even though many authors (e.g. Desai (1973); Shah and Desai (1981); Wolfstetter (1982); van der Ploeg (1983, 1984, 1987); Foley (2003)) have extended Goodwin's model by relaxing one or more of his restrictive assumptions, it is noteworthy that only a few appear to recognise, let alone address, the obvious problem of the state variables attaining unrealistic values. One exception is Blatt (1983, pp. 210–1), who suggests the introduction of a floor level for net investment as a possible solution. Another author who has noted this limitation of Goodwin's model is Peter Flaschel who, in a series of solo and collaborative publications over many years, has proposed solutions such as the inclusion of money and a state sector which operates fiscal policy and various other imaginative extensions and additional nonlinearities (Flaschel, 1987, 1993; Flaschel *et al*, 1997; Chiarella and Flaschel, 2000). Finally, Desai *et al* (2006), after pointing out the problem, make two modifications which resolve it: first, they make the real-wage equation (the Phillips curve) nonlinear; second, they relax Goodwin's assumption that all profits are always reinvested, proposing instead that the rate of investment is a function of the gap between the actual profit rate and some reservation rate.

In practice, of course, both state variables in the model vary over a far smaller range than $[0, 1]$. For example, in the UK, over the period 1855–1997, employment exceeded 99 per cent of the labour force in only twelve years, of which nine were during a world-war.

The highest employment proportion of the period was 99.7 per cent, occurring in 1916. Conversely, even in the trough of the Depression of the 1930s, the employment proportion fell below 85% only once, in 1932. With the exception of five or six years in the 1930s, the average minimum employment proportion over the last century-and-a-half is approximately 89 per cent. The wage-share series contained a strong upward trend for most of this period. It reached around 70 per cent of national income in the mid-1970s, having risen from around 50 per cent over the preceding century. This trend has since been reversed: by the 1990s, the wage share of national income had fallen back to around 55 per cent. However, within individual cycles, fluctuations in this variable have been much smaller. For example, from 1920 to 1925 it fell by just 5 per cent (from 65 per cent to 60 per cent). Conversely, in the 1970s, with wage growth outstripping that of productivity, the change was of a similar magnitude, growing from 68 per cent at the turn of the decade to 74 per cent, its peak value for the whole period, in 1975. Although such changes seem to be small in a numerical sense, they are economically significant. Prior to 1974, declining profit share was the major cause of declining profit rates in European and other advanced capitalist economies (Glyn *et al*, 1991). The 'squeeze' on profits was

sufficiently severe to herald the end of the so-called 'Golden Age': the resulting contraction of investment (and disinvestment) were in turn responsible for the rising unemployment, recession and erosion of workers' strength towards the end of the 1970s and into the 1980s (Glyn and Sutcliffe, 1972; Glyn *et al*, 1991). Given this stylised behaviour of the two state variables, it is clear they should be modelled such that neither can exceed unity, i.e., their cycles should occur only within the *unit square* in the *u-v phase plane*.

A second problem with Goodwin's model is its symmetry. Although the model is nonlinear, its solution cycles are almost symmetric, at least for the relatively small cycle amplitudes which are economically realistic. That is, for each state variable, its expansionary and contractionary phases are of similar duration, while the height of its peaks (relative to the equilibrium value) is similar to the depth of its troughs. However, it has long been recognised that the business cycle is asymmetric, with the upswing typically prolonged and gradual, whilst the downswing tends to be sharper and more sudden. For example, Keynes (1936) wrote in the *General Theory*, 'There is, however, another characteristic of what we call the trade cycle which our explanation must cover; namely, the phenomenon of the crisis — the fact that the substitution of a downward for an upward tendency often takes place suddenly and violently, whereas there is, as a rule, no such sharp turning point when an upward is substituted for a downward tendency.' (p. 214)

Recent work on business-cycle asymmetries has both redefined the characterisation of asymmetries (so that at least four different types may be distinguished) and developed rigorous new techniques for detecting their presence. In this paper we concentrate on two forms of business-cycle asymmetry which, following Sichel (1993), we refer to as *steepness* and *deepness*. Business-cycle steepness refers to the tendency, just noted, of the expansion, or upswing, to be of longer duration than the contraction, or downswing, which must therefore be necessarily steeper in average gradient. Business-cycle deepness concerns the relative distances from trend of the cycle's peak and trough: in a deep cycle, the trough is deeper relative to trend than the peak is tall.

Early observations on business-cycle asymmetry mostly concerned steepness and were underpinned by the dating of business-cycle turning points, work pioneered in the United States by Mitchell and Burns at the National Bureau of Economic Research (NBER) (Mitchell, 1927; Mitchell and Burns, 1938; Burns and Mitchell, 1946). Considering the 31 complete cycles identified by the NBER for the US economy over the period 1854–1991, for example, the average duration of the upswing was 35 months, whilst that for the downswing was just 18 months.² More recently, this form of asymmetry has been redefined by Neftçi (1984) in terms of the transition probabilities of finite-order Markov processes, and has been empirically investigated using this framework by Neftçi (1984); Falk (1986); Sichel (1989) and Rothman (1991), who variously find some evidence of significant asymmetry in variables such as unemployment and real GNP. Using nonlinear time-scale transformations to relate business-cycle time

scales to calendar time, Stock (1987) finds evidence in support of the steepness hypothesis, whilst Ramsey and Rothman (1996), testing for the time reversibility of series, reach a similar conclusion. Finally, DeLong and Summers (1986); Sichel (1993) and others estimate the skewness of a series' first differences to assess the degree of its steepness asymmetry. Although De-Long and Summers (1986) find little evidence of business-cycle steepness, Sichel (1993), Speight (1997) and Speight and McMillan (1998) reject the zero-skewness null hypothesis for a number of the series they consider, including unemployment.³

Sichel (1993) also suggests using skewness, this time of a series' levels, to measure its deepness. Using this test, he finds 'fairly strong evidence' of this form of asymmetry in (US) unemployment, as well as industrial production. Speight (1997) and Speight and McMillan (1998) also find evidence of asymmetric deepness, or its opposite, *tallness*, in some of the series they consider, including employment (which is tall) and unemployment (which is deep).

Empirical evidence on business-cycle steepness and deepness is mixed. Moreover, whilst some researchers have investigated the asymmetry properties of real wages and real output, and employment and unemployment, existing work has considered neither the wage share nor the employment proportion. The existing empirical evidence is sufficiently strong, however, to suggest that symmetry should not be taken as an *a priori* assumption in business-cycle models. That is, models should be able to generate asymmetric solution cycles even in the special cases, in which the outcomes are symmetric, are later shown to be empirically adequate. The new, Goodwin-type model presented in this paper, besides solving the unit-square problem mentioned above, is able to generate such cycles.

Our innovation is new to economics and is analogous to the concept of prey *carrying capacity* developed in the literature on biological predator-prey models. The carrying capacity of a species is determined by environmental factors: a population of a prey species will not grow beyond a certain size — the carrying capacity — even in the absence of any predator species. In our economic model, the carrying capacities are the natural limits which the wage share and employment proportion cannot exceed, i.e., unity for both variables. In our first modification of Goodwin's model, we add simple terms to the two state equations which reflect this fact. However high is employment, and the consequent strength of workers, wage-share growth will be tempered as the profit share declines. Similarly, however low is the wage share (however high the profit share), employment growth will be tempered as employment proportion approaches unity. This first extension is sufficient to prevent either state variable from exceeding unity, thereby overcoming the first problem with Goodwin's model.

In our second extension, we further modify the model such that the dynamics of wage-share and employmentproportion growth can be *tuned* as the levels of the two state variables fluctuate over the range [0, 1]. In the resulting model, we can distinguish between the contradictory effects that growth in each

variable has on its own further growth. A higher value of the wage share means that workers are more powerful and better able to further increase their share. But it also means that, since the profit share is consequently lower, capitalists are more likely to respond with pricing and other strategies to defend their share. This will make it harder for workers to achieve their objective. Similarly, given differential inter-class savings ratios, a higher employment proportion is likely to imply a higher level of aggregate demand, so adding a further impetus to employment growth. But a high employment proportion also poses potential problems (for employers) of labour recruitment and retention, which will restrain this growth. By adjusting the tuning parameters, which control the relative magnitudes of these four effects, we can adjust the symmetry properties of the resulting solution cycles. Moreover, in implicitly modelling aggregate demands effects, our modified model addresses a common theoretical criticism of Goodwin's model: that it ignores this key aspect of market-capitalist economies.

The structure of the paper is as follows. We begin, in section 2, by briefly reviewing Goodwin's growth-cycle model, before discussing some of its limitations. In section 3 we introduce our two extensions to Goodwin's model. In this section we discuss the modified model's basis in the biological literature and briefly outline its economic justification. We investigate our new model's mathematical properties in section 4, showing analytically that, like Goodwin's model, all solution trajectories (within the unit square in phase-variable space) are closed. In section 5 we revisit the model's economics, discussing in more detail its economic underpinnings. Since the coupled ordinary differential equations arising in our model do not admit a closed-form integral solution, we use numerical methods to investigate the model's period and other characteristics. We explain the methodology underlying these numerical simulations in section 6 and report simulation results in section 7. We conclude in section 8.

2. GOODWIN'S MODEL

Goodwin's original growth-cycle model is defined by the eight equations:

$$a = a_0 \exp(\alpha t), \quad \alpha > 0; \quad (2.1)$$

$$n = n_0 \exp(\beta t), \quad \beta > 0; \quad (2.2)$$

$$\sigma = \frac{k}{q}; \quad (2.3)$$

$$l = \frac{q}{a}; \quad (2.4)$$

$$u = \frac{wl}{q} = \frac{w}{a}; \quad (2.5)$$

$$v = \frac{l}{n}; \quad (2.6)$$

$$\dot{k} = (1-u)q; \quad (2.7)$$

$$\frac{\dot{w}}{w} = -\gamma + \rho v, \quad \gamma > 0, \quad \rho > 0. \quad (2.8)$$

Here, a is productivity (output per worker), hence α is productivity growth; n is the labour force, hence β is labour force growth; k is the total capital stock; q is real output; l is employment; σ is the capital-output ratio; u is workers' share of national income; v is the employment proportion; w is the real wage. The parameters γ and ρ are respectively the (negative of the) intercept and gradient of the *Phillips curve* discussed below.

Equations (2.1) and (2.2) capture Goodwin's assumptions of steady (disembodied) technical progress and steady growth of the labour force, respectively. Equation (2.3) defines the output-capital ratio, assumed to be fixed. Equation (2.4) gives the level of employment, which ensures a constant output-capital ratio. Equations (2.5) and (2.6) define the state variables, workers' share of national income and the employment rate, respectively. Equation (2.7) describes the behavioural assumptions of capitalists and workers: all wages are consumed and all profits are saved and reinvested. Finally, equation (2.8) is Goodwin's linear approximation of the Phillips curve relationship (Phillips, 1958), which captures the assumption that real wages rise in the neighbourhood of full employment.

Equations (2.1)–(2.8) reduce to a pair of coupled ordinary differential equations describing the mutually dependent growth of the variables u and v :

$$\frac{\dot{u}}{u} = -(\alpha + \gamma) + \rho v, \quad (2.9)$$

$$\frac{\dot{v}}{v} = \frac{1}{\sigma} - (\alpha + \beta) - \frac{1}{\sigma} u, \quad (2.10)$$

in which a dot superscript refers to differentiation with respect to time. The solution trajectories of the system (2.9)–(2.10) comprise a family of closed cycles in the u - v phase plane (Wolfstetter, 1982; Flaschel, 1984) which enclose the *stationary point*, where $\dot{u} = \dot{v} = 0$,

$$u^* = 1 - \sigma(\alpha + \beta),$$

$$v^* = \frac{\alpha + \gamma}{\rho}.$$

Accordingly, (u^*, v^*) is a *centre*. It is not possible to obtain a closed-form expression for the cycle's period, T , in the full nonlinear system. However, close to the stationary point, a consideration of the eigenvalues of the linearised system reveals that, in the asymptotic limit $u \rightarrow u^*$, $v \rightarrow v^*$, the period is approximated by

$$T_{\text{approx}} = \frac{2\pi}{\left[(\alpha + \gamma)(1/\sigma) - (\alpha + \beta) \right]^{1/2}} = 2\pi \sqrt{\frac{\sigma}{\rho u^* v^*}} \quad (2.11)$$

The appeal of Goodwin's model lies in its simplicity and its ability to model an important characteristic of capitalist economies. However, the model is unrealistic for several reasons. Fundamentally, and of primary interest to us, is the possibility that solution trajectories may stray outside $U = [0, 1] \times [0, 1]$ in the u - v phase plane. Since state variables u and v are defined as ratios, all possible solutions of (2.9)–(2.10) should satisfy $(u(t), v(t)) \in U$ for all $t \geq 0$. Goodwin's model, however, does not capture this characteristic.

A second feature of Goodwin's model is its neutral stability property, manifest in the existence of the closed-cycle solution trajectories discussed above. Here, solution trajectories may stray outside U , even as a result of a negative shock to either u or v . To see this, suppose that T_0 is a closed trajectory passing through $(u_0, v_0) \in U$, and that $T_0 \in U$ for all $t > 0$. Consider a shock to the economy which reduces either $u(t)$ or $v(t)$ when these are beneath their equilibrium values. The perturbed solution will then be a different, larger closed trajectory, say T_1 , for which there now may be a time t_1 such that $T_1 \notin U$. But no structural parameters have altered and the model's stationary point remains unchanged. So, irrespective of the asymptotic stability of any more realistic model, it must contain barrier functions to preclude such behaviour.

A third and related characteristic of Goodwin's model is the degree to which its cycles are symmetric. The oscillations of both phase variables in the model are approximately sinusoidal. Consequently, both variables exhibit symmetry in terms of both business-cycle *deepness* and *steepness*. First, peaks and troughs are equidistant from the equilibrium trend in Goodwin's model. Second, expansions and contractions are of equal duration and absolute gradient. The solution trajectories of the model exhibit symmetry of two other forms, as well. First, peaks and troughs are equally *rounded*.⁴ Second, the model is approximately symmetric in the way in which stochastic shocks affect its solution cycles: a shock during the expansionary phase which causes the employment proportion to rise by (say) one percentage point is approximately equivalent to a shock in the contractionary phase which causes this proportion to fall by a percentage point since both cause the economy to shift to new cycles which are very close to each other. This form of potential asymmetry is different to the asymmetric shock-response investigated and

modelled by Potter (1994, 1995), who is concerned with the persistence of shocks, i.e. the speed at which a series returns to trend following a shock. Since Goodwin's model has neutral stability, any shock to its solution trajectories is perfectly persistent.

The preceding discussion has highlighted some of the limitations of Goodwin's model and we illustrate them further with results from simulating the model in section 7. The Goodwin model has other economic shortcomings which we have not discussed here. Most important of these is its neglect of aggregate demand considerations, in particular its assumptions that all profits are reinvested and of full capital utilisation.⁵ Although the modifications we propose do not directly address these weaknesses, our resulting model nevertheless behaves in a way more realistic than Goodwin's as far as they are concerned, as we show in section 5. In the next two sections, we first present two extensions to Goodwin's model which suggest a way of overcoming the limitations discussed, before analysing the new model's mathematical properties.

3. A NEW MODEL

A major source of the Goodwin model's unrealism is the fact that wage share growth or contraction depends only upon employment proportion: the current distribution of income does not directly affect its future development. At the same time, growth or contraction of the employment proportion is determined only by the wage share: the current degree of employment/unemployment has no effect on how that ratio will change. This would suggest that any modified model should include terms in positive powers of u in the expression for \dot{u}/u , and terms in positive powers of v in the expression for \dot{v}/v . Moreover, not only do we require the state variables u and v to be constrained within U as discussed above, we would also expect the growth in these variables to decelerate they approach unity. Thus, in our modified system we require $u \rightarrow 0$ as $u \rightarrow 1$ and $v \rightarrow 0$ as $v \rightarrow 1$. This we achieve by the incorporation of so-called *barrier functions*.

Our modified model takes the form:

$$\frac{\dot{u}}{u} = [-(\alpha' + \gamma') + \rho'v] f_1(u), \quad (3.1)$$

$$\frac{\dot{v}}{v} = \left[\frac{1}{\sigma'} - (\alpha' + \beta') - \frac{1}{\sigma'}u \right] f_2(v), \quad (3.2)$$

where, for the model to be economically realistic, $f_1(u) \rightarrow 0$ as $u \rightarrow 1$, with $f_1(u) > 0$ for $u < 1$, and $f_2(v) \rightarrow 0$ as $v \rightarrow 1$, with $f_2(v) > 0$ for $v < 1$. The primed parameters α' , β' , γ' , ρ' and σ' correspond to α , β , γ , ρ and σ of equations (2.9) and (2.10), however their meaning in the new model is subtly altered. We discuss this in more detail below, in section 5.

We then choose the following forms for f_1 and f_2 :

$$f_1(u) = \kappa_1(1 - u), \quad \kappa_1 > 0, \quad (3.3)$$

$$f_2(v) = \kappa_2(1 - v), \quad \kappa_2 > 0, \quad (3.4)$$

where κ_1 and κ_2 are constants.

Our model thus becomes:

$$\frac{\dot{u}}{u} = \kappa_1(1 - u) \left[-(\alpha' + \gamma') + \rho'v \right], \quad (3.5)$$

$$\frac{\dot{v}}{v} = \kappa_2(1 - v) \left[\frac{1}{\sigma'} - (\alpha' + \beta') - \frac{1}{\sigma'}u \right]. \quad (3.6)$$

The forms for f_1 and f_2 given by equations (3.3) and (3.4)—and the resulting equations (3.5)–(3.6)—are sufficient to guarantee that solution trajectories remain within the economically realistic unit square of the u - v phase plane. The term $(1 - u)$ in f_1 is the profit share. As it approaches zero (as the wage share approaches unity), growth in the wage share must also slow to zero. Thus, wage share can never exceed unity. Similarly, the term $(1 - v)$ in f_2 is the unemployment proportion. As it tends towards zero, employment growth will also approach zero, preventing the employment proportion ever exceeding unity.

We can note here that our innovation is analogous to the concept of *carrying capacity* in ecological models. In ecological predator-prey models, the growth in species population will depend upon, not only inter-species interaction, but also intra-species interaction and wider environmental resources. Both of these additional factors will act to limit population growth to the so-called environmental carrying capacity. This is the stable equilibrium density to which a species' population tends as a result of intraspecific competition. It is called a carrying capacity because it represents the population size that the resources of the environment can just maintain, or carry (Begon *et al.*, 1996). Each of our expressions f_1 and f_2 is analogous to the Verhulst-Pearl logistic form, $g = r(1 - v/C)$, where $g = \dot{v}/v$ is the species growth rate, r is the maximum species growth rate which would (hypothetically) obtain when the species population, v , is zero, and C is the species carrying capacity. Since, in our economic model the limiting value (or carrying capacity) of both state variables is unity, we have $C \equiv 1$.⁶

Thus far we have modified Goodwin's model such that rising u will dampen the growth in u and rising v will dampen the growth in v . In reality, however, the effect of both variables on their own subsequent growth is contradictory. There exist both positive and negative feedback effects of wage-share level on wage-share growth, and of employment-proportion level on

employment-proportion growth. The forms of f_1 and f_2 given by equations (3.3) and (3.4) capture only one pole of these contradictory relationships. In our second extension to Goodwin's model, we modify f_1 and f_2 in such a way that both poles, that is, both positive and negative feedback effects, are captured. The resulting model has a rich economic interpretation, which we will discuss in more detail in section 5, below, and moreover generates interesting economic outcomes, including asymmetric cycles, as we show in section 7.

Our new forms of f_1 and f_2 are:

$$f_1(u) = \kappa_1 u^{\mu_1} (1-u)^{\eta_1}, \quad \kappa_1 > 0, \mu_1 \geq 0, \eta_1 > 0, \quad (3.7)$$

$$f_2(v) = \kappa_2 v^{\mu_2} (1-v)^{\eta_2}, \quad \kappa_2 > 0, \mu_2 \geq 0, \eta_2 > 0. \quad (3.8)$$

The tuning parameters—indices μ_1 , η_1 , μ_2 and η_2 and multipliers κ_1 and κ_2 in (3.7) and (3.8)—allow the generation of asymmetric cycles since they cause the dynamics (or propagation) of the cycle to vary over its constituent stages. Consider, for example, expression (3.7). When u is small, say $0 < u = \epsilon \ll 1$, we have

$$f_1(\epsilon) \sim \kappa_1 \epsilon^{\mu_1} - \kappa_1 \eta_1 \epsilon^{\mu_1+1} + O(\epsilon^{\mu_1+2}), \quad \epsilon \rightarrow 0,$$

so that only the parameter μ_1 is influential in determining the value of $f_1(u)$ and hence the rate of change of the wage share. On the other hand, with u close to unity, say $0 < 1 - u = \epsilon \ll 1$, we have

$$f_1(1-\epsilon) \sim \kappa_1 \epsilon^{\eta_1} - \kappa_1 \mu_1 \epsilon^{\eta_1+1} + O(\epsilon^{\eta_1+2}), \quad \epsilon \rightarrow 0,$$

and hence it is now the parameter η_1 which plays the influential role in determining the rate of change of the wage share. Analogous considerations apply to $f_2(v)$ in (3.8).

The model thus becomes

$$\frac{\dot{u}}{u} = \kappa_1 \mu^{\mu_1} (1-u)^{\eta_1} [-(\alpha' + \gamma') + \rho'v] \quad (3.9)$$

$$\frac{\dot{v}}{v} = \kappa_2 v^{\mu_2} (1-v)^{\eta_2} \left[\frac{1}{\sigma'} - (\alpha' + \beta' - \frac{1}{\sigma'}u) \right]. \quad (3.10)$$

Setting $\mu_1 = \eta_1 = \mu_2 = \eta_2 = 0$ and $\kappa_1 = \kappa_2 = 1$, and $\alpha' = \alpha$, $\beta' = \beta$, etc., our modified model reduces to Goodwin's original specification.

4 STATIONARY POINTS, STABILITY AND THE CYCLE'S PERIOD

In this section we analyse the model and show that its solution trajectories describe closed curves. We consider first the linearised system, then show that the stationary point not on the boundary of U is, in fact, a nonlinear centre. We go on to consider the cycle's period.

We write equations (3.9)–(3.10) as

$$\dot{u} = \kappa_1 u^{\eta_1+1} (1-u)^{\eta_1} [-a_1 + b_1 v], \tag{4.1}$$

$$\dot{v} = \kappa_2 v^{\eta_2+1} (1-v)^{\eta_2} [-a_2 + b_2 u], \tag{4.2}$$

in which

$$\begin{aligned} a_1 &= \alpha' + \gamma' &> 0 \\ b_1 &= \rho' &> 0 \\ a_2 &= 1/\sigma' - (\alpha' + \beta') &> 0 \\ b_2 &= 1/\sigma' &> 0 \end{aligned}$$

wherein the restrictions $0 < a_i < b_i$ apply for $i = 1, 2$.

In both Goodwin's model and our model, the u - and v -axes are barriers which cannot be crossed by solution trajectories in the u - v phase plane. The barrier functions $f_1(u)$ and $f_2(v)$ introduced above prevent solution trajectories from crossing the other two sides of the unit square U . To see this in the full model, one can use (4.1) and (4.2) to show that, when either u or v is near to unity, say $0 < 1 - u, 1 - v = \epsilon \ll 1$, we have

$$\begin{aligned} \dot{u} &= \kappa_1 (b_1 v - \alpha_1) \epsilon^{\eta_1} + O(\epsilon^{\eta_1+1}), & v \in [0,1] \\ \dot{v} &= \kappa_2 (b_2 u - \alpha_2) \epsilon^{\eta_2} + O(\epsilon^{\eta_2+1}), & u \in [0,1] \end{aligned}$$

Thus for $\eta_1, \eta_2 > 0$, $\dot{u}, \dot{v} \rightarrow 0$ as $\epsilon \rightarrow 0$ and the barrier functions therefore play their role satisfactorily. Expressions can be derived for \ddot{u} and \ddot{v} to determine the relevant decelerations as the barriers are approached, but these require a consideration of balances of different indices in order to determine the most dominant effects as $\epsilon \rightarrow 0$; as such they are particularly complicated and do not add to our discussion. Note that, in Goodwin's model, in which all indices are zero, a similar argument gives $\dot{u} \rightarrow \kappa_1 (b_1 v - \alpha_1)$ and $\dot{v} \rightarrow \kappa_2 (b_2 u - \alpha_2)$ as $\epsilon \rightarrow 0$, confirming that trajectories cross the boundary of U with a non-vanishing velocity when $v \neq \alpha_1/b_1$ and $u \neq \alpha_2/b_2$ respectively.

Turning now to a consideration of stability, we set $\dot{u} = 0$ and $\dot{v} = 0$ in equations (4.1) and (4.2) to obtain five stationary points: each of the vertices of U (all of which are saddle points) and the point $(u^*, v^*) = (\alpha_2/b_2, \alpha_1/b_1)$.

Linearising about the stationary point (u^*, v^*) internal to U via the Galilean transformations $u = U + u^*$ and $v = V + v^*$, we obtain

$$U = \kappa_1 b_1 [U + u^*]^{\mu_1 + 1} [1 - u^* - U]^{\eta_1} V, \quad (4.3)$$

$$V = -\kappa_2 b_2 [V + v^*]^{\mu_2 + 1} [1 - v^* - V]^{\eta_2} U, \quad (4.4)$$

For which the Jacobian matrix is

$$J \equiv \frac{\partial(\dot{U}, \dot{V})}{\partial(U, V)}(u^*, v^*) = \begin{bmatrix} 0 & \kappa_1 b_1 (u^*)^{\eta_1 + 1} (1 - u^*)^{\eta_1} \\ -\kappa_2 b_2 (v^*)^{\eta_2 + 1} (1 - v^*)^{\eta_2} & 0 \end{bmatrix}.$$

The characteristic equation for the eigenvalues of $J, |J - \lambda I| = 0$ therefore gives

$$\lambda^2 + \kappa_1 \kappa_2 b_1 b_2 (u^*)^{\mu_1 + 1} (1 - u^*)^{\eta_1} (v^*)^{\mu_2 + 1} (1 - v^*)^{\eta_2} = 0,$$

wherein the second term on the left-hand side is always positive. Thus we have $\lambda = \pm i\Omega$, where

$$\begin{aligned} \Omega^2 &= \kappa_1 \kappa_2 b_1 b_2 (u^*)^{\mu_1 + 1} (1 - u^*)^{\eta_1} (v^*)^{\mu_2 + 1} (1 - v^*)^{\eta_2} \\ &= \kappa_1 \kappa_2 a_1 a_2 (u^*)^{\mu_1} (1 - u^*)^{\eta_1} (v^*)^{\mu_2} (1 - v^*)^{\eta_2}. \end{aligned} \quad (4.5)$$

Since the eigenvalues λ are purely imaginary, the stationary point of the linearised system is a centre, and so the full nonlinear system could have (Arrowsmith and Place, 1982) either a centre or a (stable or unstable) focus at (u^*, v^*) . The cycles are in fact closed, as we now show.

Dividing equation (4.2) by equation (4.1) we obtain

$$\frac{dv}{du} = \frac{\kappa_2 v^{1+\mu_2} (1-v)^{\eta_2} (a_2 - b_2 u)}{\kappa_1 u^{1+\mu_1} (1-u)^{\eta_1} (-a_1 + b_1 v)}, \quad (4.6)$$

which is a separable equation with first integral

$$\int_0^u \frac{a_2 - b_2 u'}{\kappa_1 (u')^{1+\mu_1} (1-u')^{\eta_1}} du' + \int_0^v \frac{a_1 - b_1 v'}{\kappa_2 (v')^{1+\mu_2} (1-v')^{\eta_2}} dv' = \ln K, \text{ a constant.} \quad (4.7)$$

Now define in turn

$$\begin{aligned} p_i &\equiv (\kappa_i, \mu_i, \eta_i; a_i, b_i), \\ h(\xi; p_i) &\equiv \frac{a_i - b_i \xi}{\kappa_i \xi^{1+\mu_i} (1-\xi)^{\eta_i}} d\xi, \quad 0 < a_i < b_i \\ H(\xi; p_i) &\equiv \int_0^\xi h(\xi'; p_i) d\xi', \quad 0 \leq \xi \leq 1 \\ G(\xi; p_i) &\equiv \exp\{H(\xi; p_i)\}, \end{aligned}$$

so that (4.7) yields

$$G(u; p_1) G(v; p_2) = K \tag{4.11}$$

We therefore have

$$\frac{\partial G(\xi; p_i)}{\partial \xi} = \exp\{H(\xi; p_i)\} \frac{\partial H(\xi; p_i)}{\partial \xi} = G(\xi; p_i) h(\xi; p_i), \tag{4.12}$$

wherein the factor $G(\xi; p_i)$ is clearly non-negative for all $\xi \in [0, 1]$. We note from (4.8) and (4.9) that,

$$H(\xi; p_i) \sim -\frac{a_i}{\kappa_i \mu_i} \xi^{-\mu_i} + \frac{\eta_i - b_i}{\kappa_i (1 - \mu_i)} \xi^{1-\mu_i} + O(\xi^{2-\mu_i}), \quad \xi \rightarrow 0^+$$

which, since $\mu_i > 0$, tends to $-\infty$ as $\xi \rightarrow 0^+$. Hence, by (4.10), $G(\xi; p_i) \rightarrow 0^+$ as $\xi \rightarrow 0^+$ and, by (4.8), G therefore increases monotonically from zero to a maximum value $G_{\max} = \exp\{H(a_i/b_i; p_i)\}$ in $\xi \in [0, a_i/b_i]$, wherafter it decreases monotonically (but not necessarily back down to zero) in $\xi \in (a_i/b_i, 1]$. Hence G has a unique maximum in the unit interval and so (u, v) pairs in U satisfying (4.11) lie on closed trajectories: the full nonlinear system is indeed a centre.

It is recognised that, for this particular system of coupled differential equations, the ability to prove the existence of a (global) nonlinear centre is a consequence of the separability of (4.6). Whilst such a global proof is impossible in the more general case, whether or not (u, v) constitutes a local centre can be determined via an asymptotic, algorithmic stability analysis as per Davies and James (1966). In particular, when both the numerator and the denominator on the right-hand side in (4.6) are homogeneous polynomials of second degree in u and v , a more specific method due to Bautin (Davies and James, 1966, p.193) is appropriate.

The duration of the business cycle is of interest to economists and policy-makers. With this in mind we consider the cycle's period. In the neighbourhood of the stationary point $(u, v) = (u^*, v^*) = (a_2/b_2, a_1/b_1)$, the linearised system is

$$\dot{U} = \kappa_1 b_1 [u^*]^{\mu_1-1} [1-u^*]^{\eta_1} V, \tag{4.13}$$

$$\dot{V} = -\kappa_2 b_2 [v^*]^{\mu_2-1} [1-v^*]^{\eta_2} U. \tag{4.14}$$

This system has solution

$$\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} \kappa_1 \cos(\Omega t + \omega) \\ \kappa_2 \sin(\Omega t + \omega) \end{bmatrix}$$

where k_1 , k_2 and ω are constants which depend upon the parameters in equations (4.13) and (4.14), and Ω is defined in equation (4.5), above. It therefore follows that, close to the equilibrium point (u^*, v^*) , the cycle's period T is approximated by

$$T_{\text{approx}} = \frac{2\pi}{\Omega} = \frac{2\pi}{\left\{ \kappa_1 \kappa_2 a_1 a_2 (u^*)^{\mu_1} (1-u^*)^{\eta_1} (v^*)^{\mu_2} (1-v^*)^{\eta_2} \right\}^{1/2}}. \quad (4.15)$$

It is clear from equation (4.15) that T_{approx} increases with increasing values of the powers μ_1 , η_1 , μ_2 and η_2 and decreases with increasing κ_1 and κ_2 . When we have $\mu_2 = \eta_2 = \mu_1 = \eta_1 = 0$ and $\kappa_1 = \kappa_2 = 1$, the expression gives the period of the approximation to the original Goodwin cycle, as expected.

Equation (4.15) is only an approximation to T , valid close to the stationary point. As we move further away from (u, v) , the approximation becomes less accurate. For general values of the parameters, it is not possible to obtain explicit closed-form solutions for $u(t)$ and $v(t)$ and we necessarily resort to numerical integration of (3.9)–(3.10); in particular, this will also be used to investigate more thoroughly the cycle's period away from (u^*, v^*) . But before this investigation, we turn to the economic interpretation of our modification of Goodwin's model.

5. ECONOMICS OF THE NEW WORLD

Our modified model has a rich economic interpretation, reflecting many aspects of reality not captured by the original specification, including: questions of social power and control, aggregate demand, and variable rates of capital utilisation, productivity growth and labour-force growth.

The expression $f_1(u) = \kappa_1 u^{\mu_1} (1-u)^{\eta_1}$ in equations (3.1) and (3.5) reflects the fact that changes in distributional shares are influenced by the level of the wage share, and its inverse, the profit share and, as such, f_1 captures aspects of capitalist-economy macro-dynamics not present in Goodwin's model. Moreover, this expression directly limits workers' share of national income such that it cannot rise above 100 per cent. In Goodwin's model, real-wage growth is determined solely by workers' strength in the labour market, proxied by the employment proportion. In real economies, other factors influence the level of the real wage and the aggregate distribution of income into wages and profits. Here we distinguish two of these factors, which have contradictory effects. First, firms' price-setting behaviour is influenced by both target and actual distributional shares. The lower the profit share of national income (i.e., the higher the wage share), the more likely firms are to respond to wage growth with price increases. We can associate this effect with the $(1-u)^{\eta_1}$ term in $f_1(u)$. Second, we can think of the phase variable u as measuring more than the distribution of income; it is also a measure of the relative social power of the two classes. Money, or share of income, can be used not only to purchase

consumption goods and services, but in addition to finance trade unions, political parties and other forms of organisation. In short, workers attempt to use their share of national income to further increase this share. We can associate this effect with the u^{μ_1} term in $f_1(u)$. The tuning parameters, the multiplicative term κ_1 and the powers μ_1 and η_1 measure the way in which the dynamic modelling of social power, including firms' pricing behaviour, unfolds as the underlying distributional shares change. One would expect their values to be determined empirically.

In the expression for employment proportion growth (equations (3.2) and (3.6)), the term $f_2(v) = \kappa_2 v^{\mu_2} (1-v)^{\eta_2}$ reflects the fact that this variable cannot exceed unity. More generally, it captures the extent to which employment and unemployment proportion levels affect the way in which these variables *change*. The function $f_2(v)$, like the function $f_1(u)$ in the wage-share growth equation, captures two effects. First, in Goodwin's specification, growing employment brakes further output and employment growth indirectly, due to its effect on profits. We term this the *distributional* or *profit-squeeze* limit to employment. However, there is in addition a direct limit to employment and employment growth. As the employment proportion grows, the unemployed proportion of the workforce correspondingly falls, it becomes increasingly difficult for employers to fill vacancies due to sectoral, skill and geographical rigidities and also their own increasing search costs. We term this the *natural* or *physical* limit to employment.⁷ Second, as we briefly noted in section 2, in his original model, Goodwin does not consider the role of aggregate or effective demand (the *realisation problem* in Marxian terminology). He simply assumes that all wages are consumed and all profits are reinvested, whilst the capital-output ratio remains constant. More realistically, investment will depend not only upon profits and the profit share, but also upon the rate of utilisation. In our specification, we cannot model this effect explicitly. However, we do capture it implicitly: as we show below, utilisation initially rises (capital-output ratio falls) with rising employment, and the *rate* of employment proportion growth also (initially) rises as its own level rises. Thus, there is an association between utilisation and employment growth. This effective demand effect is associated with the term v^{μ_2} in $f_2(v)$. The tuning parameters κ_2 , μ_2 and η_2 measure the strengths of this limiting factor and of the multiplier effect. As for κ_1 , μ_1 and η_1 in the wage-growth equation, one would expect their values to be determined empirically.

The parameters, α' , β' , γ' , ρ' and σ' , in our equations (3.5) and (3.6) do not correspond exactly to α , β , γ , ρ and σ in Goodwin's specification (equations (2.9) and (2.10)). In Goodwin's model, these parameters are equal to the true values of, respectively, productivity growth, labour-force growth, (minus one times) Phillips curve intercept and slope, and capital-output ratio, which are all assumed to be constant. In our model these economic variables do in fact vary.

The corresponding primed parameters in our model can be thought of as *base values*; the true values will depend also upon the tuning parameter and the stage of the cycle, i.e., upon u and/or v .

$$\frac{\dot{u}}{u} = \frac{\dot{w}}{w} - \frac{\dot{a}}{a}. \quad (5.1)$$

Then from equations (2.4) and (2.6), we have,

$$v = \frac{q}{an}, \quad (5.2)$$

so that we can write employment-proportion growth as

$$\frac{\dot{v}}{v} = \frac{\dot{q}}{q} - \frac{\dot{a}}{a} - \frac{\dot{n}}{n}. \quad (5.3)$$

It follows from equations (3.9) and (5.1) and from (3.10) and (5.3) that we must have

$$\frac{\dot{w}}{w} - \frac{\dot{a}}{a} = \kappa_1 u^{\mu_1} (1-u)^n [-(\alpha' + \gamma') + \rho'v], \quad (5.4)$$

$$\frac{\dot{q}}{q} - \frac{\dot{a}}{a} - \frac{\dot{n}}{n} = \kappa_2 v^{\mu_2} (1-v)^{n_2} \left[\frac{1}{\sigma'} - (\alpha' + \beta') \frac{1}{\sigma' u} \right]. \quad (5.5)$$

We can then see from equations (5.4) and (5.5) how productivity growth and the other structural variables vary over the business cycle. Denoting actual values by a carat, we obtain, from equation (5.4),

$$\hat{\alpha} = \kappa_1 u^{\mu_1} (1-u)^n \alpha' = f_1(u) \alpha', \quad (5.6)$$

$$\hat{\gamma} = \kappa_1 u^{\mu_1} (1-u)^n \gamma' = f_1(u) \gamma', \quad (5.7)$$

$$\hat{\rho} = \kappa_1 u^{\mu_1} (1-u)^n \rho' = f_1(u) \rho'. \quad (5.8)$$

And from equation (5.5) we obtain,

$$\hat{\sigma} = \frac{1}{\kappa_2 v^{\mu_2} (1-v)^{n_2}} \sigma' = \frac{1}{f_2(v)} \sigma', \quad (5.9)$$

$$\widehat{\alpha + \beta} = \kappa_2 v^{\mu_2} (1-v)^{n_2} (\alpha' + \beta') = f_2(v) (\alpha' + \beta'). \quad (5.10)$$

Considering first true productivity growth $\hat{\alpha}$, this variable is given explicitly as a function of u (equation (5.6)). This function takes value zero at $u = 0$ and $u = 1$, and is positive for $0 < u < 1$, in which it has a unique maximum. Differentiating equation (5.6) with respect to u we obtain

$$\frac{d\hat{\alpha}}{du} = \kappa_1 \alpha u^{\mu_1 - 1} (1 - u)^{\eta_1 - 1} [\mu_1 - (\mu_1 + \eta_1)u]. \quad (5.11)$$

Since the terms preceding the square brackets in equation (5.11) are always positive, it is easy to see that true productivity growth is maximised relative to u at $u = \mu_1 / (\mu_1 + \eta_1)$.

Thus equation (5.6), which expresses productivity growth as a function of wage share has an inverted- U shape. This relationship is congruent with economic theory. The positive relationship between productivity growth rates and wage share at lower values of the latter variable is consistent with a number of theoretical frameworks. For example, in Schumpeterian or evolutionary theory, firms which innovate are able to acquire market power and hence become more profitable. As wages and wage share rise, such innovating firms are better able to survive the consequent fall in the profit share (Geroski *et al.*, 1993). That is, as wage share rises, competitive pressures in the economy intensify, with innovating, productivity-enhancing firms more likely to survive the enhanced process of creative destruction (Kleinknecht, 1998). Alternatively, within an endogenous growth framework, the higher the wage share, the more likely profit-maximising firms are to invest in technology likely to reduce labour costs through raising productivity. However, beyond a certain level of wage share, one would expect this variable's relationship with the rate of productivity growth to become negative. As profits are 'squeezed', expectations of future profits will also be revised downwards and firms will tend to be both less willing and less able to make new investments. Since most productivity-enhancing innovations can only be realised through investment in new technology, productivity growth will be lowered in this situation.

Productivity growth also appears in equation (5.10), but cannot be separated from labour-force growth in this expression. This term, $\widehat{\alpha + \beta}$, measures the rate at which the employment proportion declines, *ceteris paribus*, as a result of the combined effects of productivity growth and labour-force growth.

The function (5.10) has a similar form to (5.6): $\widehat{\alpha + \beta}$ is zero at $v = 0$ and $v = 1$ and is positive between these values. It has a unique maximum at $v = \mu_2 / (\mu_2 + \eta_2)$, which we can interpret as the employment proportion at which the displacement (both actual and potential) of existing workers is maximised.

Again, the inverted- U -shaped form of (5.10) is congruent with economic theory. Considering first productivity growth, if we take employment proportion to be a proxy for the level of effective demand in the economy,⁸ the the positive association between these two variables, $\hat{\alpha}$ on the one hand, and productivity