

A Dynamical Model of Business-Cycle Asymmetries: Extending Goodwin

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ABSTRACT

A rarely noted, economically unrealistic feature of Goodwin's (1967; 1972) celebrated 'growth-cycle' model is that its state variables — the wage share of output and the employment proportion — can exceed unity. We propose a novel extension of the two-variable dynamical system which ensures that its solutions remain within the economically feasible region, i.e. the unit square of the wage-share-employment-proportion phase plane. In a further extension, we obtain a model which, besides possessing a richer economic interpretation than the original, is able to generate asymmetric solution cycles. We use numerical techniques to investigate the new model's properties; in particular, we examine business-cycle deepness and steepness.

1. INTRODUCTION

The Goodwin growth cycle (Goodwin, 1967, 1972) is a macroeconomic model, analogous to the Lotka-Volterra predator-prey model, which describes population dynamics of competing species (Lotka, 1956; Volterra, 1931a,b, 1937). In the Goodwin model, the dynamic interaction is between the distribution of income and the proportion of the workforce in employment. In this model, the wage share of national income is the predator variable and the employment proportion is the prey.

Goodwin himself described his model as 'starkly schematized and hence quite unrealistic' (Goodwin, 1967, p.54; 1972, p.442). Nonetheless, despite its limitations, the model has proved to be remarkably popular and has inspired a huge literature. Part of the model's enduring appeal lies in its simplicity; in particular, the elegance with which it illustrates the cyclical relationship between income distribution and employment in a dynamic capitalist economy over the course of the business cycle. Now, four decades on, the model still inspires new contributions to the economics literature, particularly that part of it concerned with dynamic modelling (e.g., Manfredi and Fanti (2004)).

The state variables in Goodwin's model are expressed in the form of ratios: wage *share* of national income and employment *proportion*. Yet, internal to the original model, there are no forces which prevent either of these variables from exceeding unity; an outcome which is, by definition, unrealistic. In real economies, neither the wage share of national income nor the employment proportion can exceed unity. Even though many authors (e.g. Desai (1973); Shah and Desai (1981); Wolfstetter (1982); van der Ploeg (1983, 1984, 1987); Foley (2003)) have extended Goodwin's model by relaxing one or more of his restrictive assumptions, it is noteworthy that only a few appear to recognise, let alone address, the obvious problem of the state variables attaining unrealistic values. One exception is Blatt (1983, pp. 210–1), who suggests the introduction of a floor level for net investment as a possible solution. Another author who has noted this limitation of Goodwin's model is Peter Flaschel who, in a series of solo and collaborative publications over many years, has proposed solutions such as the inclusion of money and a state sector which operates fiscal policy and various other imaginative extensions and additional nonlinearities (Flaschel, 1987, 1993; Flaschel *et al*, 1997; Chiarella and Flaschel, 2000). Finally, Desai *et al* (2006), after pointing out the problem, make two modifications which resolve it: first, they make the real-wage equation (the Phillips curve) nonlinear; second, they relax Goodwin's assumption that all profits are always reinvested, proposing instead that the rate of investment is a function of the gap between the actual profit rate and some reservation rate.

In practice, of course, both state variables in the model vary over a far smaller range than $[0, 1]$. For example, in the UK, over the period 1855–1997, employment exceeded 99 per cent of the labour force in only twelve years, of which nine were during a world-war.

The highest employment proportion of the period was 99.7 per cent, occurring in 1916. Conversely, even in the trough of the Depression of the 1930s, the employment proportion fell below 85% only once, in 1932. With the exception of five or six years in the 1930s, the average minimum employment proportion over the last century-and-a-half is approximately 89 per cent. The wage-share series contained a strong upward trend for most of this period. It reached around 70 per cent of national income in the mid-1970s, having risen from around 50 per cent over the preceding century. This trend has since been reversed: by the 1990s, the wage share of national income had fallen back to around 55 per cent. However, within individual cycles, fluctuations in this variable have been much smaller. For example, from 1920 to 1925 it fell by just 5 per cent (from 65 per cent to 60 per cent). Conversely, in the 1970s, with wage growth outstripping that of productivity, the change was of a similar magnitude, growing from 68 per cent at the turn of the decade to 74 per cent, its peak value for the whole period, in 1975. Although such changes seem to be small in a numerical sense, they are economically significant. Prior to 1974, declining profit share was the major cause of declining profit rates in European and other advanced capitalist economies (Glyn *et al*, 1991). The 'squeeze' on profits was

sufficiently severe to herald the end of the so-called 'Golden Age': the resulting contraction of investment (and disinvestment) were in turn responsible for the rising unemployment, recession and erosion of workers' strength towards the end of the 1970s and into the 1980s (Glyn and Sutcliffe, 1972; Glyn *et al*, 1991). Given this stylised behaviour of the two state variables, it is clear they should be modelled such that neither can exceed unity, i.e., their cycles should occur only within the *unit square* in the *u-v phase plane*.

A second problem with Goodwin's model is its symmetry. Although the model is nonlinear, its solution cycles are almost symmetric, at least for the relatively small cycle amplitudes which are economically realistic. That is, for each state variable, its expansionary and contractionary phases are of similar duration, while the height of its peaks (relative to the equilibrium value) is similar to the depth of its troughs. However, it has long been recognised that the business cycle is asymmetric, with the upswing typically prolonged and gradual, whilst the downswing tends to be sharper and more sudden. For example, Keynes (1936) wrote in the *General Theory*, 'There is, however, another characteristic of what we call the trade cycle which our explanation must cover; namely, the phenomenon of the crisis — the fact that the substitution of a downward for an upward tendency often takes place suddenly and violently, whereas there is, as a rule, no such sharp turning point when an upward is substituted for a downward tendency.' (p. 214)

Recent work on business-cycle asymmetries has both redefined the characterisation of asymmetries (so that at least four different types may be distinguished) and developed rigorous new techniques for detecting their presence. In this paper we concentrate on two forms of business-cycle asymmetry which, following Sichel (1993), we refer to as *steepness* and *deepness*. Business-cycle steepness refers to the tendency, just noted, of the expansion, or upswing, to be of longer duration than the contraction, or downswing, which must therefore be necessarily steeper in average gradient. Business-cycle deepness concerns the relative distances from trend of the cycle's peak and trough: in a deep cycle, the trough is deeper relative to trend than the peak is tall.

Early observations on business-cycle asymmetry mostly concerned steepness and were underpinned by the dating of business-cycle turning points, work pioneered in the United States by Mitchell and Burns at the National Bureau of Economic Research (NBER) (Mitchell, 1927; Mitchell and Burns, 1938; Burns and Mitchell, 1946). Considering the 31 complete cycles identified by the NBER for the US economy over the period 1854–1991, for example, the average duration of the upswing was 35 months, whilst that for the downswing was just 18 months.² More recently, this form of asymmetry has been redefined by Neftçi (1984) in terms of the transition probabilities of finite-order Markov processes, and has been empirically investigated using this framework by Neftçi (1984); Falk (1986); Sichel (1989) and Rothman (1991), who variously find some evidence of significant asymmetry in variables such as unemployment and real GNP. Using nonlinear time-scale transformations to relate business-cycle time

scales to calendar time, Stock (1987) finds evidence in support of the steepness hypothesis, whilst Ramsey and Rothman (1996), testing for the time reversibility of series, reach a similar conclusion. Finally, DeLong and Summers (1986); Sichel (1993) and others estimate the skewness of a series' first differences to assess the degree of its steepness asymmetry. Although De-Long and Summers (1986) find little evidence of business-cycle steepness, Sichel (1993), Speight (1997) and Speight and McMillan (1998) reject the zero-skewness null hypothesis for a number of the series they consider, including unemployment.³

Sichel (1993) also suggests using skewness, this time of a series' levels, to measure its deepness. Using this test, he finds 'fairly strong evidence' of this form of asymmetry in (US) unemployment, as well as industrial production. Speight (1997) and Speight and McMillan (1998) also find evidence of asymmetric deepness, or its opposite, *tallness*, in some of the series they consider, including employment (which is tall) and unemployment (which is deep).

Empirical evidence on business-cycle steepness and deepness is mixed. Moreover, whilst some researchers have investigated the asymmetry properties of real wages and real output, and employment and unemployment, existing work has considered neither the wage share nor the employment proportion. The existing empirical evidence is sufficiently strong, however, to suggest that symmetry should not be taken as an *a priori* assumption in business-cycle models. That is, models should be able to generate asymmetric solution cycles even if the special cases, in which the outcomes are symmetric, are later shown to be empirically adequate. The new, Goodwin-type model presented in this paper, besides solving the unit-square problem mentioned above, is able to generate such cycles.

Our innovation is new to economics and is analogous to the concept of prey *carrying capacity* developed in the literature on biological predator-prey models. The carrying capacity of a species is determined by environmental factors: a population of a prey species will not grow beyond a certain size — the carrying capacity — even in the absence of any predator species. In our economic model, the carrying capacities are the natural limits which the wage share and employment proportion cannot exceed, i.e., unity for both variables. In our first modification of Goodwin's model, we add simple terms to the two state equations which reflect this fact. However high is employment, and the consequent strength of workers, wage-share growth will be tempered as the profit share declines. Similarly, however low is the wage share (however high the profit share), employment growth will be tempered as employment proportion approaches unity. This first extension is sufficient to prevent either state variable from exceeding unity, thereby overcoming the first problem with Goodwin's model.

In our second extension, we further modify the model such that the dynamics of wage-share and employmentproportion growth can be *tuned* as the levels of the two state variables fluctuate over the range [0, 1]. In the resulting model, we can distinguish between the contradictory effects that growth in each

variable has on its own further growth. A higher value of the wage share means that workers are more powerful and better able to further increase their share. But it also means that, since the profit share is consequently lower, capitalists are more likely to respond with pricing and other strategies to defend their share. This will make it harder for workers to achieve their objective. Similarly, given differential inter-class savings ratios, a higher employment proportion is likely to imply a higher level of aggregate demand, so adding a further impetus to employment growth. But a high employment proportion also poses potential problems (for employers) of labour recruitment and retention, which will restrain this growth. By adjusting the tuning parameters, which control the relative magnitudes of these four effects, we can adjust the symmetry properties of the resulting solution cycles. Moreover, in implicitly modelling aggregate demands effects, our modified model addresses a common theoretical criticism of Goodwin's model: that it ignores this key aspect of market-capitalist economies.

The structure of the paper is as follows. We begin, in section 2, by briefly reviewing Goodwin's growth-cycle model, before discussing some of its limitations. In section 3 we introduce our two extensions to Goodwin's model. In this section we discuss the modified model's basis in the biological literature and briefly outline its economic justification. We investigate our new model's mathematical properties in section 4, showing analytically that, like Goodwin's model, all solution trajectories (within the unit square in phase-variable space) are closed. In section 5 we revisit the model's economics, discussing in more detail its economic underpinnings. Since the coupled ordinary differential equations arising in our model do not admit a closed-form integral solution, we use numerical methods to investigate the model's period and other characteristics. We explain the methodology underlying these numerical simulations in section 6 and report simulation results in section 7. We conclude in section 8.

2. GOODWIN'S MODEL

Goodwin's original growth-cycle model is defined by the eight equations:

$$a = a_0 \exp(\alpha t), \quad \alpha > 0; \quad (2.1)$$

$$n = n_0 \exp(\beta t), \quad \beta > 0; \quad (2.2)$$

$$\sigma = \frac{k}{q}; \quad (2.3)$$

$$l = \frac{q}{a}; \quad (2.4)$$

$$u = \frac{wl}{q} = \frac{w}{a}; \quad (2.5)$$

$$v = \frac{l}{n}; \quad (2.6)$$

$$\dot{k} = (1-u)q; \quad (2.7)$$

$$\frac{\dot{w}}{w} = -\gamma + \rho v, \quad \gamma > 0, \quad \rho > 0. \quad (2.8)$$

Here, a is productivity (output per worker), hence α is productivity growth; n is the labour force, hence β is labour force growth; k is the total capital stock; q is real output; l is employment; σ is the capital-output ratio; u is workers' share of national income; v is the employment proportion; w is the real wage. The parameters γ and ρ are respectively the (negative of the) intercept and gradient of the *Phillips curve* discussed below.

Equations (2.1) and (2.2) capture Goodwin's assumptions of steady (disembodied) technical progress and steady growth of the labour force, respectively. Equation (2.3) defines the output-capital ratio, assumed to be fixed. Equation (2.4) gives the level of employment, which ensures a constant output-capital ratio. Equations (2.5) and (2.6) define the state variables, workers' share of national income and the employment rate, respectively. Equation (2.7) describes the behavioural assumptions of capitalists and workers: all wages are consumed and all profits are saved and reinvested. Finally, equation (2.8) is Goodwin's linear approximation of the Phillips curve relationship (Phillips, 1958), which captures the assumption that real wages rise in the neighbourhood of full employment.

Equations (2.1)–(2.8) reduce to a pair of coupled ordinary differential equations describing the mutually dependent growth of the variables u and v :

$$\frac{\dot{u}}{u} = -(\alpha + \gamma) + \rho v, \quad (2.9)$$

$$\frac{\dot{v}}{v} = \frac{1}{\sigma} - (\alpha + \beta) - \frac{1}{\sigma} u, \quad (2.10)$$

in which a dot superscript refers to differentiation with respect to time. The solution trajectories of the system (2.9)–(2.10) comprise a family of closed cycles in the u - v phase plane (Wolfstetter, 1982; Flaschel, 1984) which enclose the *stationary point*, where $\dot{u} = \dot{v} = 0$,

$$u^* = 1 - \sigma(\alpha + \beta),$$

$$v^* = \frac{\alpha + \gamma}{\rho}.$$

Accordingly, (u^*, v^*) is a *centre*. It is not possible to obtain a closed-form expression for the cycle's period, T , in the full nonlinear system. However, close to the stationary point, a consideration of the eigenvalues of the linearised system reveals that, in the asymptotic limit $u \rightarrow u^*$, $v \rightarrow v^*$, the period is approximated by

$$T_{\text{approx}} = \frac{2\pi}{\left[(\alpha + \gamma)(1/\sigma) - (\alpha + \beta) \right]^{1/2}} = 2\pi \sqrt{\frac{\sigma}{\rho u^* v^*}} \quad (2.11)$$

The appeal of Goodwin's model lies in its simplicity and its ability to model an important characteristic of capitalist economies. However, the model is unrealistic for several reasons. Fundamentally, and of primary interest to us, is the possibility that solution trajectories may stray outside $U = [0, 1] \times [0, 1]$ in the u - v phase plane. Since state variables u and v are defined as ratios, all possible solutions of (2.9)–(2.10) should satisfy $(u(t), v(t)) \in U$ for all $t \geq 0$. Goodwin's model, however, does not capture this characteristic.

A second feature of Goodwin's model is its neutral stability property, manifest in the existence of the closed-cycle solution trajectories discussed above. Here, solution trajectories may stray outside U , even as a result of a negative shock to either u or v . To see this, suppose that T_0 is a closed trajectory passing through $(u_0, v_0) \in U$, and that $T_0 \in U$ for all $t > 0$. Consider a shock to the economy which reduces either $u(t)$ or $v(t)$ when these are beneath their equilibrium values. The perturbed solution will then be a different, larger closed trajectory, say T_1 , for which there now may be a time t_1 such that $T_1 \notin U$. But no structural parameters have altered and the model's stationary point remains unchanged. So, irrespective of the asymptotic stability of any more realistic model, it must contain barrier functions to preclude such behaviour.

A third and related characteristic of Goodwin's model is the degree to which its cycles are symmetric. The oscillations of both phase variables in the model are approximately sinusoidal. Consequently, both variables exhibit symmetry in terms of both business-cycle *deepness* and *steepness*. First, peaks and troughs are equidistant from the equilibrium trend in Goodwin's model. Second, expansions and contractions are of equal duration and absolute gradient. The solution trajectories of the model exhibit symmetry of two other forms, as well. First, peaks and troughs are equally *rounded*.⁴ Second, the model is approximately symmetric in the way in which stochastic shocks affect its solution cycles: a shock during the expansionary phase which causes the employment proportion to rise by (say) one percentage point is approximately equivalent to a shock in the contractionary phase which causes this proportion to fall by a percentage point since both cause the economy to shift to new cycles which are very close to each other. This form of potential asymmetry is different to the asymmetric shock-response investigated and

modelled by Potter (1994, 1995), who is concerned with the persistence of shocks, i.e. the speed at which a series returns to trend following a shock. Since Goodwin's model has neutral stability, any shock to its solution trajectories is perfectly persistent.

The preceding discussion has highlighted some of the limitations of Goodwin's model and we illustrate them further with results from simulating the model in section 7. The Goodwin model has other economic shortcomings which we have not discussed here. Most important of these is its neglect of aggregate demand considerations, in particular its assumptions that all profits are reinvested and of full capital utilisation.⁵ Although the modifications we propose do not directly address these weaknesses, our resulting model nevertheless behaves in a way more realistic than Goodwin's as far as they are concerned, as we show in section 5. In the next two sections, we first present two extensions to Goodwin's model which suggest a way of overcoming the limitations discussed, before analysing the new model's mathematical properties.

3. A NEW MODEL

A major source of the Goodwin model's unrealism is the fact that wage share growth or contraction depends only upon employment proportion: the current distribution of income does not directly affect its future development. At the same time, growth or contraction of the employment proportion is determined only by the wage share: the current degree of employment/unemployment has no effect on how that ratio will change. This would suggest that any modified model should include terms in positive powers of u in the expression for \dot{u}/u , and terms in positive powers of v in the expression for \dot{v}/v . Moreover, not only do we require the state variables u and v to be constrained within U as discussed above, we would also expect the growth in these variables to decelerate they approach unity. Thus, in our modified system we require $u \rightarrow 0$ as $u \rightarrow 1$ and $v \rightarrow 0$ as $v \rightarrow 1$. This we achieve by the incorporation of so-called *barrier functions*.

Our modified model takes the form:

$$\frac{\dot{u}}{u} = [-(\alpha' + \gamma') + \rho'v] f_1(u), \quad (3.1)$$

$$\frac{\dot{v}}{v} = \left[\frac{1}{\sigma'} - (\alpha' + \beta') - \frac{1}{\sigma'}u \right] f_2(v), \quad (3.2)$$

where, for the model to be economically realistic, $f_1(u) \rightarrow 0$ as $u \rightarrow 1$, with $f_1(u) > 0$ for $u < 1$, and $f_2(v) \rightarrow 0$ as $v \rightarrow 1$, with $f_2(v) > 0$ for $v < 1$. The primed parameters α' , β' , γ' , ρ' and σ' correspond to α , β , γ , ρ and σ of equations (2.9) and (2.10), however their meaning in the new model is subtly altered. We discuss this in more detail below, in section 5.

We then choose the following forms for f_1 and f_2 :

$$f_1(u) = \kappa_1(1-u), \quad \kappa_1 > 0, \quad (3.3)$$

$$f_2(v) = \kappa_2(1-v), \quad \kappa_2 > 0, \quad (3.4)$$

where κ_1 and κ_2 are constants.

Our model thus becomes:

$$\frac{\dot{u}}{u} = \kappa_1(1-u) \left[-(\alpha' + \gamma') + \rho'v \right], \quad (3.5)$$

$$\frac{\dot{v}}{v} = \kappa_2(1-v) \left[\frac{1}{\sigma'} - (\alpha' + \beta') - \frac{1}{\sigma'}u \right]. \quad (3.6)$$

The forms for f_1 and f_2 given by equations (3.3) and (3.4)—and the resulting equations (3.5)–(3.6)—are sufficient to guarantee that solution trajectories remain within the economically realistic unit square of the u - v phase plane. The term $(1-u)$ in f_1 is the profit share. As it approaches zero (as the wage share approaches unity), growth in the wage share must also slow to zero. Thus, wage share can never exceed unity. Similarly, the term $(1-v)$ in f_2 is the unemployment proportion. As it tends towards zero, employment growth will also approach zero, preventing the employment proportion ever exceeding unity.

We can note here that our innovation is analogous to the concept of *carrying capacity* in ecological models. In ecological predator-prey models, the growth in species population will depend upon, not only inter-species interaction, but also intra-species interaction and wider environmental resources. Both of these additional factors will act to limit population growth to the so-called environmental carrying capacity. This is the stable equilibrium density to which a species' population tends as a result of intraspecific competition. It is called a carrying capacity because it represents the population size that the resources of the environment can just maintain, or carry (Begon *et al.*, 1996). Each of our expressions f_1 and f_2 is analogous to the Verhulst-Pearl logistic form, $g = r(1 - v/C)$, where $g = \dot{v}/v$ is the species growth rate, r is the maximum species growth rate which would (hypothetically) obtain when the species population, v , is zero, and C is the species carrying capacity. Since, in our economic model the limiting value (or carrying capacity) of both state variables is unity, we have $C \equiv 1$.⁶

Thus far we have modified Goodwin's model such that rising u will dampen the growth in u and rising v will dampen the growth in v . In reality, however, the effect of both variables on their own subsequent growth is contradictory. There exist both positive and negative feedback effects of wage-share level on wage-share growth, and of employment-proportion level on

employment-proportion growth. The forms of f_1 and f_2 given by equations (3.3) and (3.4) capture only one pole of these contradictory relationships. In our second extension to Goodwin's model, we modify f_1 and f_2 in such a way that both poles, that is, both positive and negative feedback effects, are captured. The resulting model has a rich economic interpretation, which we will discuss in more detail in section 5, below, and moreover generates interesting economic outcomes, including asymmetric cycles, as we show in section 7.

Our new forms of f_1 and f_2 are:

$$f_1(u) = \kappa_1 u^{\mu_1} (1-u)^{\eta_1}, \quad \kappa_1 > 0, \mu_1 \geq 0, \eta_1 > 0, \quad (3.7)$$

$$f_2(v) = \kappa_2 v^{\mu_2} (1-v)^{\eta_2}, \quad \kappa_2 > 0, \mu_2 \geq 0, \eta_2 > 0. \quad (3.8)$$

The tuning parameters—indices μ_1 , η_1 , μ_2 and η_2 and multipliers κ_1 and κ_2 in (3.7) and (3.8)—allow the generation of asymmetric cycles since they cause the dynamics (or propagation) of the cycle to vary over its constituent stages. Consider, for example, expression (3.7). When u is small, say $0 < u = \epsilon \ll 1$, we have

$$f_1(\epsilon) \sim \kappa_1 \epsilon^{\mu_1} - \kappa_1 \eta_1 \epsilon^{\mu_1+1} + O(\epsilon^{\mu_1+2}), \quad \epsilon \rightarrow 0,$$

so that only the parameter μ_1 is influential in determining the value of $f_1(u)$ and hence the rate of change of the wage share. On the other hand, with u close to unity, say $0 < 1 - u = \epsilon \ll 1$, we have

$$f_1(1-\epsilon) \sim \kappa_1 \epsilon^{\eta_1} - \kappa_1 \mu_1 \epsilon^{\eta_1+1} + O(\epsilon^{\eta_1+2}), \quad \epsilon \rightarrow 0,$$

and hence it is now the parameter η_1 which plays the influential role in determining the rate of change of the wage share. Analogous considerations apply to $f_2(v)$ in (3.8).

The model thus becomes

$$\frac{\dot{u}}{u} = \kappa_1 \mu^{\eta_1} (1-u)^{\eta_1} [-(\alpha' + \gamma') + \rho'v] \quad (3.9)$$

$$\frac{\dot{v}}{v} = \kappa_2 v^{\mu_2} (1-v)^{\eta_2} \left[\frac{1}{\sigma'} - (\alpha' + \beta' - \frac{1}{\sigma'}u) \right]. \quad (3.10)$$

Setting $\mu_1 = \eta_1 = \mu_2 = \eta_2 = 0$ and $\kappa_1 = \kappa_2 = 1$, and $\alpha' = \alpha$, $\beta' = \beta$, etc., our modified model reduces to Goodwin's original specification.

4 STATIONARY POINTS, STABILITY AND THE CYCLE'S PERIOD

In this section we analyse the model and show that its solution trajectories describe closed curves. We consider first the linearised system, then show that the stationary point not on the boundary of U is, in fact, a nonlinear centre. We go on to consider the cycle's period.

We write equations (3.9)–(3.10) as

$$\dot{u} = \kappa_1 u^{\eta_1+1} (1-u)^{\eta_1} [-a_1 + b_1 v], \tag{4.1}$$

$$\dot{v} = \kappa_2 v^{\eta_2+1} (1-v)^{\eta_2} [-a_2 + b_2 u], \tag{4.2}$$

in which

$$\begin{aligned} a_1 &= \alpha' + \gamma' &> 0 \\ b_1 &= \rho' &> 0 \\ a_2 &= 1/\sigma' - (\alpha' + \beta') &> 0 \\ b_2 &= 1/\sigma' &> 0 \end{aligned}$$

wherein the restrictions $0 < a_i < b_i$ apply for $i = 1, 2$.

In both Goodwin's model and our model, the u - and v -axes are barriers which cannot be crossed by solution trajectories in the u - v phase plane. The barrier functions $f_1(u)$ and $f_2(v)$ introduced above prevent solution trajectories from crossing the other two sides of the unit square U . To see this in the full model, one can use (4.1) and (4.2) to show that, when either u or v is near to unity, say $0 < 1 - u, 1 - v = \epsilon \ll 1$, we have

$$\begin{aligned} \dot{u} &= \kappa_1 (b_1 v - a_1) \epsilon^{\eta_1} + O(\epsilon^{\eta_1+1}), & v \in [0,1] \\ \dot{v} &= \kappa_2 (b_2 u - a_2) \epsilon^{\eta_2} + O(\epsilon^{\eta_2+1}), & u \in [0,1] \end{aligned}$$

Thus for $\eta_1, \eta_2 > 0$, $\dot{u}, \dot{v} \rightarrow 0$ as $\epsilon \rightarrow 0$ and the barrier functions therefore play their role satisfactorily. Expressions can be derived for \ddot{u} and \ddot{v} to determine the relevant decelerations as the barriers are approached, but these require a consideration of balances of different indices in order to determine the most dominant effects as $\epsilon \rightarrow 0$; as such they are particularly complicated and do not add to our discussion. Note that, in Goodwin's model, in which all indices are zero, a similar argument gives $\dot{u} \rightarrow \kappa_1 (b_1 v - a_1)$ and $\dot{v} \rightarrow \kappa_2 (b_2 u - a_2)$ as $\epsilon \rightarrow 0$, confirming that trajectories cross the boundary of U with a non-vanishing velocity when $v \neq a_1/b_1$ and $u \neq a_2/b_2$ respectively.

Turning now to a consideration of stability, we set $\dot{u} = 0$ and $\dot{v} = 0$ in equations (4.1) and (4.2) to obtain five stationary points: each of the vertices of U (all of which are saddle points) and the point $(u^*, v^*) = (a_2/b_2, a_1/b_1)$.

Linearising about the stationary point (u^*, v^*) internal to U via the Galilean transformations $u = U + u^*$ and $v = V + v^*$, we obtain

$$U = \kappa_1 b_1 [U + u^*]^{\mu_1 + 1} [1 - u^* - U]^{\eta_1} V, \quad (4.3)$$

$$V = -\kappa_2 b_2 [V + v^*]^{\mu_2 + 1} [1 - v^* - V]^{\eta_2} U, \quad (4.4)$$

For which the Jacobian matrix is

$$J \equiv \frac{\partial(\dot{U}, \dot{V})}{\partial(U, V)}(u^*, v^*) = \begin{bmatrix} 0 & \kappa_1 b_1 (u^*)^{\eta_1 + 1} (1 - u^*)^{\eta_1} \\ -\kappa_2 b_2 (v^*)^{\eta_2 + 1} (1 - v^*)^{\eta_2} & 0 \end{bmatrix}.$$

The characteristic equation for the eigenvalues of $J, |J - \lambda I| = 0$ therefore gives

$$\lambda^2 + \kappa_1 \kappa_2 b_1 b_2 (u^*)^{\mu_1 + 1} (1 - u^*)^{\eta_1} (v^*)^{\mu_2 + 1} (1 - v^*)^{\eta_2} = 0,$$

wherein the second term on the left-hand side is always positive. Thus we have $\lambda = \pm i\Omega$, where

$$\begin{aligned} \Omega^2 &= \kappa_1 \kappa_2 b_1 b_2 (u^*)^{\mu_1 + 1} (1 - u^*)^{\eta_1} (v^*)^{\mu_2 + 1} (1 - v^*)^{\eta_2} \\ &= \kappa_1 \kappa_2 a_1 a_2 (u^*)^{\mu_1} (1 - u^*)^{\eta_1} (v^*)^{\mu_2} (1 - v^*)^{\eta_2}. \end{aligned} \quad (4.5)$$

Since the eigenvalues λ are purely imaginary, the stationary point of the linearised system is a centre, and so the full nonlinear system could have (Arrowsmith and Place, 1982) either a centre or a (stable or unstable) focus at (u^*, v^*) . The cycles are in fact closed, as we now show.

Dividing equation (4.2) by equation (4.1) we obtain

$$\frac{dv}{du} = \frac{\kappa_2 v^{1+\mu_2} (1-v)^{\eta_2} (a_2 - b_2 u)}{\kappa_1 u^{1+\mu_1} (1-u)^{\eta_1} (-a_1 + b_1 v)}, \quad (4.6)$$

which is a separable equation with first integral

$$\int_0^u \frac{a_2 - b_2 u'}{\kappa_1 (u')^{1+\mu_1} (1-u')^{\eta_1}} du' + \int_0^v \frac{a_1 - b_1 v'}{\kappa_2 (v')^{1+\mu_2} (1-v')^{\eta_2}} dv' = \ln K, \text{ a constant.} \quad (4.7)$$

Now define in turn

$$\begin{aligned} p_i &\equiv (\kappa_i, \mu_i, \eta_i; a_i, b_i), \\ h(\xi; p_i) &\equiv \frac{a_i - b_i \xi}{\kappa_i \xi^{1+\mu_i} (1-\xi)^{\eta_i}} d\xi, \quad 0 < a_i < b_i \\ H(\xi; p_i) &\equiv \int_0^\xi h(\xi'; p_i) d\xi', \quad 0 \leq \xi \leq 1 \\ G(\xi; p_i) &\equiv \exp\{H(\xi; p_i)\}, \end{aligned}$$

so that (4.7) yields

$$G(u; p_1) G(v; p_2) = K \tag{4.11}$$

We therefore have

$$\frac{\partial G(\xi; p_i)}{\partial \xi} = \exp\{H(\xi; p_i)\} \frac{\partial H(\xi; p_i)}{\partial \xi} = G(\xi; p_i) h(\xi; p_i), \tag{4.12}$$

wherein the factor $G(\xi; p_i)$ is clearly non-negative for all $\xi \in [0, 1]$. We note from (4.8) and (4.9) that,

$$H(\xi; p_i) \sim -\frac{a_i}{\kappa_i \mu_i} \xi^{-\mu_i} + \frac{\eta_i - b_i}{\kappa_i (1 - \mu_i)} \xi^{1-\mu_i} + O(\xi^{2-\mu_i}), \quad \xi \rightarrow 0^+$$

which, since $\mu_i > 0$, tends to $-\infty$ as $\xi \rightarrow 0^+$. Hence, by (4.10), $G(\xi; p_i) \rightarrow 0^+$ as $\xi \rightarrow 0^+$ and, by (4.8), G therefore increases monotonically from zero to a maximum value $G_{\max} = \exp\{H(a_i/b_i; p_i)\}$ in $\xi \in [0, a_i/b_i]$, wherafter it decreases monotonically (but not necessarily back down to zero) in $\xi \in (a_i/b_i, 1]$. Hence G has a unique maximum in the unit interval and so (u, v) pairs in U satisfying (4.11) lie on closed trajectories: the full nonlinear system is indeed a centre.

It is recognised that, for this particular system of coupled differential equations, the ability to prove the existence of a (global) nonlinear centre is a consequence of the separability of (4.6). Whilst such a global proof is impossible in the more general case, whether or not (u, v) constitutes a local centre can be determined via an asymptotic, algorithmic stability analysis as per Davies and James (1966). In particular, when both the numerator and the denominator on the right-hand side in (4.6) are homogeneous polynomials of second degree in u and v , a more specific method due to Bautin (Davies and James, 1966, p.193) is appropriate.

The duration of the business cycle is of interest to economists and policy-makers. With this in mind we consider the cycle's period. In the neighbourhood of the stationary point $(u, v) = (u^*, v^*) = (a_2/b_2, a_1/b_1)$, the linearised system is

$$\dot{U} = \kappa_1 b_1 [u^*]^{\mu_1-1} [1-u^*]^{\eta_1} V, \tag{4.13}$$

$$\dot{V} = -\kappa_2 b_2 [v^*]^{\mu_2-1} [1-v^*]^{\eta_2} U. \tag{4.14}$$

This system has solution

$$\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} \kappa_1 \cos(\Omega t + \omega) \\ \kappa_2 \sin(\Omega t + \omega) \end{bmatrix}$$

where k_1 , k_2 and ω are constants which depend upon the parameters in equations (4.13) and (4.14), and Ω is defined in equation (4.5), above. It therefore follows that, close to the equilibrium point (u^*, v^*) , the cycle's period T is approximated by

$$T_{\text{approx}} = \frac{2\pi}{\Omega} = \frac{2\pi}{\left\{ \kappa_1 \kappa_2 a_1 a_2 (u^*)^{\mu_1} (1-u^*)^{\eta_1} (v^*)^{\mu_2} (1-v^*)^{\eta_2} \right\}^{1/2}}. \quad (4.15)$$

It is clear from equation (4.15) that T_{approx} increases with increasing values of the powers μ_1 , η_1 , μ_2 and η_2 and decreases with increasing κ_1 and κ_2 . When we have $\mu_2 = \eta_2 = \mu_1 = \eta_1 = 0$ and $\kappa_1 = \kappa_2 = 1$, the expression gives the period of the approximation to the original Goodwin cycle, as expected.

Equation (4.15) is only an approximation to T , valid close to the stationary point. As we move further away from (u, v) , the approximation becomes less accurate. For general values of the parameters, it is not possible to obtain explicit closed-form solutions for $u(t)$ and $v(t)$ and we necessarily resort to numerical integration of (3.9)–(3.10); in particular, this will also be used to investigate more thoroughly the cycle's period away from (u^*, v^*) . But before this investigation, we turn to the economic interpretation of our modification of Goodwin's model.

5. ECONOMICS OF THE NEW WORLD

Our modified model has a rich economic interpretation, reflecting many aspects of reality not captured by the original specification, including: questions of social power and control, aggregate demand, and variable rates of capital utilisation, productivity growth and labour-force growth.

The expression $f_1(u) = \kappa_1 u^{\mu_1} (1-u)^{\eta_1}$ in equations (3.1) and (3.5) reflects the fact that changes in distributional shares are influenced by the level of the wage share, and its inverse, the profit share and, as such, f_1 captures aspects of capitalist-economy macro-dynamics not present in Goodwin's model. Moreover, this expression directly limits workers' share of national income such that it cannot rise above 100 per cent. In Goodwin's model, real-wage growth is determined solely by workers' strength in the labour market, proxied by the employment proportion. In real economies, other factors influence the level of the real wage and the aggregate distribution of income into wages and profits. Here we distinguish two of these factors, which have contradictory effects. First, firms' price-setting behaviour is influenced by both target and actual distributional shares. The lower the profit share of national income (i.e., the higher the wage share), the more likely firms are to respond to wage growth with price increases. We can associate this effect with the $(1-u)^{\eta_1}$ term in $f_1(u)$. Second, we can think of the phase variable u as measuring more than the distribution of income; it is also a measure of the relative social power of the two classes. Money, or share of income, can be used not only to purchase

consumption goods and services, but in addition to finance trade unions, political parties and other forms of organisation. In short, workers attempt to use their share of national income to further increase this share. We can associate this effect with the u^{μ_1} term in $f_1(u)$. The tuning parameters, the multiplicative term κ_1 and the powers μ_1 and η_1 measure the way in which the dynamic modelling of social power, including firms' pricing behaviour, unfolds as the underlying distributional shares change. One would expect their values to be determined empirically.

In the expression for employment proportion growth (equations (3.2) and (3.6)), the term $f_2(v) = \kappa_2 v^{\mu_2} (1-v)^{\eta_2}$ reflects the fact that this variable cannot exceed unity. More generally, it captures the extent to which employment and unemployment proportion levels affect the way in which these variables *change*. The function $f_2(v)$, like the function $f_1(u)$ in the wage-share growth equation, captures two effects. First, in Goodwin's specification, growing employment brakes further output and employment growth indirectly, due to its effect on profits. We term this the *distributional* or *profit-squeeze* limit to employment. However, there is in addition a direct limit to employment and employment growth. As the employment proportion grows, the unemployed proportion of the workforce correspondingly falls, it becomes increasingly difficult for employers to fill vacancies due to sectoral, skill and geographical rigidities and also their own increasing search costs. We term this the *natural* or *physical* limit to employment.⁷ Second, as we briefly noted in section 2, in his original model, Goodwin does not consider the role of aggregate or effective demand (the *realisation problem* in Marxian terminology). He simply assumes that all wages are consumed and all profits are reinvested, whilst the capital-output ratio remains constant. More realistically, investment will depend not only upon profits and the profit share, but also upon the rate of utilisation. In our specification, we cannot model this effect explicitly. However, we do capture it implicitly: as we show below, utilisation initially rises (capital-output ratio falls) with rising employment, and the *rate* of employment proportion growth also (initially) rises as its own level rises. Thus, there is an association between utilisation and employment growth. This effective demand effect is associated with the term v^{μ_2} in $f_2(v)$. The tuning parameters κ_2 , μ_2 and η_2 measure the strengths of this limiting factor and of the multiplier effect. As for κ_1 , μ_1 and η_1 in the wage-growth equation, one would expect their values to be determined empirically.

The parameters, α' , β' , γ' , ρ' and σ' , in our equations (3.5) and (3.6) do not correspond exactly to α , β , γ , ρ and σ in Goodwin's specification (equations (2.9) and (2.10)). In Goodwin's model, these parameters are equal to the true values of, respectively, productivity growth, labour-force growth, (minus one times) Phillips curve intercept and slope, and capital-output ratio, which are all assumed to be constant. In our model these economic variables do in fact vary.

The corresponding primed parameters in our model can be thought of as *base values*; the true values will depend also upon the tuning parameter and the stage of the cycle, i.e., upon u and/or v .

$$\frac{\dot{u}}{u} = \frac{\dot{w}}{w} - \frac{\dot{a}}{a}. \quad (5.1)$$

Then from equations (2.4) and (2.6), we have,

$$v = \frac{q}{an}, \quad (5.2)$$

so that we can write employment-proportion growth as

$$\frac{\dot{v}}{v} = \frac{\dot{q}}{q} - \frac{\dot{a}}{a} - \frac{\dot{n}}{n}. \quad (5.3)$$

It follows from equations (3.9) and (5.1) and from (3.10) and (5.3) that we must have

$$\frac{\dot{w}}{w} - \frac{\dot{a}}{a} = \kappa_1 u^{\mu_1} (1-u)^n [-(\alpha' + \gamma') + \rho'v], \quad (5.4)$$

$$\frac{\dot{q}}{q} - \frac{\dot{a}}{a} - \frac{\dot{n}}{n} = \kappa_2 v^{\mu_2} (1-v)^{n_2} \left[\frac{1}{\sigma'} - (\alpha' + \beta') \frac{1}{\sigma' u} \right]. \quad (5.5)$$

We can then see from equations (5.4) and (5.5) how productivity growth and the other structural variables vary over the business cycle. Denoting actual values by a carat, we obtain, from equation (5.4),

$$\hat{\alpha} = \kappa_1 u^{\mu_1} (1-u)^n \alpha' = f_1(u) \alpha', \quad (5.6)$$

$$\hat{\gamma} = \kappa_1 u^{\mu_1} (1-u)^n \gamma' = f_1(u) \gamma', \quad (5.7)$$

$$\hat{\rho} = \kappa_1 u^{\mu_1} (1-u)^n \rho' = f_1(u) \rho'. \quad (5.8)$$

And from equation (5.5) we obtain,

$$\hat{\sigma} = \frac{1}{\kappa_2 v^{\mu_2} (1-v)^{n_2}} \sigma' = \frac{1}{f_2(v)} \sigma', \quad (5.9)$$

$$\widehat{\alpha + \beta} = \kappa_2 v^{\mu_2} (1-v)^{n_2} (\alpha' + \beta') = f_2(v) (\alpha' + \beta'). \quad (5.10)$$

Considering first true productivity growth $\hat{\alpha}$, this variable is given explicitly as a function of u (equation (5.6)). This function takes value zero at $u = 0$ and $u = 1$, and is positive for $0 < u < 1$, in which it has a unique maximum. Differentiating equation (5.6) with respect to u we obtain

$$\frac{d\hat{\alpha}}{du} = \kappa_1 \alpha u^{\mu_1 - 1} (1 - u)^{\eta_1 - 1} [\mu_1 - (\mu_1 + \eta_1)u]. \quad (5.11)$$

Since the terms preceding the square brackets in equation (5.11) are always positive, it is easy to see that true productivity growth is maximised relative to u at $u = \mu_1 / (\mu_1 + \eta_1)$.

Thus equation (5.6), which expresses productivity growth as a function of wage share has an inverted- U shape. This relationship is congruent with economic theory. The positive relationship between productivity growth rates and wage share at lower values of the latter variable is consistent with a number of theoretical frameworks. For example, in Schumpeterian or evolutionary theory, firms which innovate are able to acquire market power and hence become more profitable. As wages and wage share rise, such innovating firms are better able to survive the consequent fall in the profit share (Geroski *et al.*, 1993). That is, as wage share rises, competitive pressures in the economy intensify, with innovating, productivity-enhancing firms more likely to survive the enhanced process of creative destruction (Kleinknecht, 1998). Alternatively, within an endogenous growth framework, the higher the wage share, the more likely profit-maximising firms are to invest in technology likely to reduce labour costs through raising productivity. However, beyond a certain level of wage share, one would expect this variable's relationship with the rate of productivity growth to become negative. As profits are 'squeezed', expectations of future profits will also be revised downwards and firms will tend to be both less willing and less able to make new investments. Since most productivity-enhancing innovations can only be realised through investment in new technology, productivity growth will be lowered in this situation.

Productivity growth also appears in equation (5.10), but cannot be separated from labour-force growth in this expression. This term, $\widehat{\alpha + \beta}$, measures the rate at which the employment proportion declines, *ceteris paribus*, as a result of the combined effects of productivity growth and labour-force growth.

The function (5.10) has a similar form to (5.6): $\widehat{\alpha + \beta}$ is zero at $v = 0$ and $v = 1$ and is positive between these values. It has a unique maximum at $v = \mu_2 / (\mu_2 + \eta_2)$, which we can interpret as the employment proportion at which the displacement (both actual and potential) of existing workers is maximised.

Again, the inverted- U -shaped form of (5.10) is congruent with economic theory. Considering first productivity growth, if we take employment proportion to be a proxy for the level of effective demand in the economy,⁸ the the positive association between these two variables, $\hat{\alpha}$ on the one hand, and productivity

growth, on the other, is consistent with Schmookler's (1966) hypothesis of 'demand-pulled innovations' (Kleinknecht, 1998), which has received empirical support from Geroski and Walters (1995) and Brouwer and Kleinknecht (1999). Moreover, Kleinknecht and Naastepad (2002) cite firm-level empirical evidence that suggests a positive relationship between employment security (which is positively correlated with employment proportion), on the one hand, and innovation (Michie and Sheehan, 1999) and productivity growth (Kleinknecht *et al.*, 1997), on the other. The rationale for these findings is the enhancement of employee trust and loyalty engendered by job-security, which in turn leads to greater productivity. However, once employment proportion rises beyond a certain level, we would expect this positive relationship to be reversed. With high employment, it is likely that workers use their strength not only to secure higher wages, but also to resist, frequently tacitly, firms' attempts to increase productivity, particularly those which may result in intensified work. The clearest example of this is the productivity crisis experienced by most advanced capitalist economies in the 1970s (Glyn *et al.*, 1991).

Turning now to the impact of the employment proportion on labour-force force, it is likely that more people will be drawn into the workforce as employment grows. An important way this can happen is through labour migration, frequently facilitated through a variety of explicit and tacit government policies. In recent decades, the phenomenon of temporary and seasonal labour migration, *within* global geographical regions (in particular the European Union and Central and Eastern Europe, on the one hand, and North America, on the other) has grown in importance (OECD, 1998, 2001) and thus this factor can play a role at the shorter-range frequencies of the typical business cycle (eight years or so). Rising employment proportion might also encourage more married women to enter the labour force, however the effect of this factor over business-cycle frequencies is likely to be more marginal.

Beyond a critical value of v (in our model, $v = \mu_2 / (\mu_2 + \eta_2)$), however, we might expect this positive relationship to be reversed. First, the experience of many advanced capitalist economies in the 1960s and '70s suggests that following two decades of stable economic growth and high employment, many people began to question working patterns. The very fact that employment rates were very high meant these people were able to 'drop out' or otherwise adopt flexible working practices (on their own terms, not their employers') with very little risk to their future career prospects. A second reason for the reversal of this relationship is indirect, acting through wage levels. With high employment, workers are able to increase their own income levels. With a familial backward-bending labour-supply curve, this may result in a decline in the economically-active population.⁹

Considering now the Phillips curve, we can see from equations (5.7) and (5.8) that, in our model, although the values of the parameters $\hat{\gamma}$ and $\hat{\rho}$ vary through the cycle, their ratio does not. Thus the employment proportion at which real wage growth is zero remains constant, at $v = \hat{\gamma} / \hat{\rho}$. As the parame-

ters of the (linearised) Phillips curve vary, the 'curve' itself rotates about the point $(v, \hat{w}/w) = (\gamma'/\rho', 0)$. That is, the responsiveness of real-wage growth to the employment proportion is dependent upon the distribution of income (and of social power). The parameters $\hat{\gamma}$ and $\hat{\rho}$ reach a maximum when $u = \mu_1/(\mu_1 + \eta_1)$, i.e., this is the wage share at which the Phillips curve is steepest and real-wage growth is most responsive to the employment proportion.

The relationship between the slope of the Phillips curve and profitability has been explored by Tsakalotos (2002). He argues that the neo-Keynesian model of, say, Ball *et al.* (1988) implies a negative relationship between profitability and Phillips-curve slope. In our model, this regime corresponds to wage shares below $u = \mu_1/(\mu_1 + \eta_1)$, for which the relationship between wage share and the Phillip's curve's gradient is positive. At wage shares above this, the relationship between wage share (profitability) and Phillips-curve gradient is negative (positive) and this, Tsakalotos suggests, is consistent with a social conflict model, such as developed by Rowthorn (1977). Thus, our model is consistent with two alternative theoretical approaches to inflation: whether the cycle exists in a neo-Keynesian regime or a social conflict regime depends upon the level of the wage share. It should be emphasised that the model will not generally switch between the two regimes within a the course of a single business cycle. Over business-cycle timescales, the economy will tend to operate in a region either above or below the 'critical' value of u .¹⁰

Finally, considering true capital-output ratio, $\hat{\sigma}$, we can see from equation (5.9) that the relationship between this variable and the employment proportion is *U-shaped*. The ratio's value is minimised at $v = \mu_2/(\mu_2 + \eta_2)$. When employment proportion is below this critical value, the relationship between the two variables is negative: as employment rises, so does the capital-utilisation rate (the capital-output ratio falls). The relationship between employment proportion and growth in employment proportion is also positive below this point, which is consistent with effective demand effects and an investment function including capital utilisation on the right-hand side.¹¹ High capital-utilisation, however, tends to require a high degree of employee-flexibility in the workplace, which may involve greater work intensities, flexible working hours and so on. When the employment proportion becomes 'too high', i.e., above the critical value $v = \mu_2/(\mu_2 + \eta_2)$, our model shifts into a *conflict regime*. In this regime the relationship between capital output ratio (utilisation rate) becomes positive (negative) as workers use the rising strength associated with increased employment to oppose such workplace-flexibility.

6 NUMERICAL SIMULATIONS

As stated above, it is not possible to obtain analytical expressions for either state variable as a function of time, or for the period of the model's solution cycles, so that progress is effected by approximating the original continuous differential equations by their discrete difference-equation counterparts. Specifically, we employ a fourth-order Runge-Kutta finite-difference procedure

(Gerald and Wheatley, 1994) to integrate (3.9)–(3.10) numerically in time. We use a variable-timestep method that gives a relative error of 0.01 per cent in the simulations, as we note below in section 7.

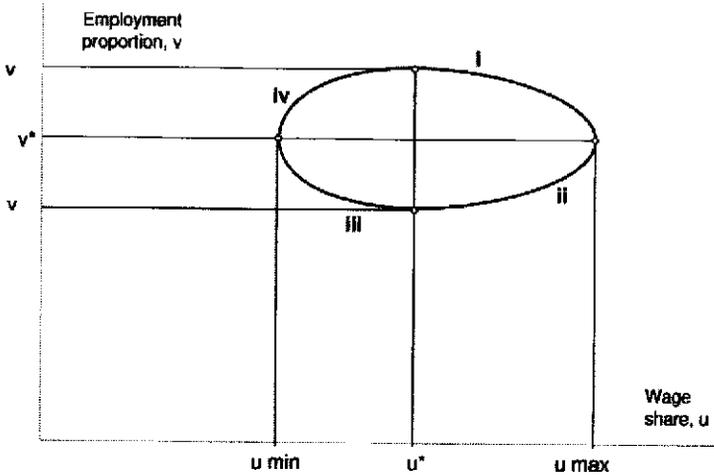


Figure 1: Annotation of points on the cycle in the u - v phase plane.

In order to consider the asymmetry of solutions, we deliberately break the cycle into four quadrants delineated by (extremal) cardinal points as per Figure 1. Given a point $(u_0, v_0) \in U$, we first use (4.11) to determine the relevant value of K , K_0 say, for the cycle on which (u_0, v_0) lies. Using these coordinates, and the parameter vectors p_1 and p_2 , calculate

$$K_0 = G(u_0; p_1) G(v_0; p_2).$$

Then, noting that the dividers of the quadrants must pass through (u^*, v^*) , which is known, we can obtain u_{min} and u_{max} as the two roots of

$$G(u; p_1) = \frac{K_0}{G(v^*; p_2)},$$

and v_{min} and v_{max} as the two roots of

$$G(v; p_2) = \frac{K_0}{G(u^*; p_1)},$$

that there are two, and only two, roots in each case is guaranteed by the aforementioned nature of G . Subsequently, integrations with a time step of $\Delta t = 0.1$ (years) are effected (N.B. anticlockwise) between consecutive extremal points around the cycle. In this way, estimates for the *quarter periods*, T_i — where $i = iv, iii, ii, i$ as per the usual trigonometric quadrant counter — can be

obtained.¹² Then T is merely the sum of these: $T = \sum_{i=1}^4 T_i$

We use these quarter-periods to calculate measures of business-cycle steepness. The wage share is contracting as the cycle moves from point (u_{\max}, v^*) to point (u_{\min}, v^*) and is expanding as the cycle returns from (u_{\min}, v^*) to (u_{\max}, v^*) . We then measure wage-share steepness by the (absolute) ratio of its average expansion gradient to its average contraction gradient, which simplifies to the ratio of the period of contraction to the period of expansion. First, we define state-variable amplitudes by $A_u \equiv u_{\max} - u_{\min}$ and $A_v \equiv v_{\max} - v_{\min}$ whereafter we define

$$u_{st} \equiv \frac{A_u / T_{ii} + T_i}{A_u / T_{iii} + T_{iv}} = \frac{T_{iii} + T_{iv}}{T_{ii} + T_i} \quad (6.1)$$

The measure u_{st} is less (greater) than unity if the wage-share contractions are steeper (less steep) than expansions; such series are called *down-steep* (*up-steep*). Similarly, the employment proportion is contracting as the cycle moves from (u^*, v_{\max}) to (u^*, v_{\min}) and expanding as the cycle moves from (u^*, v_{\min}) to (u^*, v_{\max}) . We can then define

$$u_{st} \equiv \frac{A_v / T_{iii} + T_{iv}}{A_v / T_i + T_{iv}} = \frac{T_{iv} + T_i}{T_{iii} + T_{ii}} \quad (6.2)$$

Business-cycle deepness concerns the extent to which a series' troughs are deeper relative to trend than its peaks are high. We thus use the values our state variables take at their turning points to measure their deepness. Accordingly, we define *turning-point* measures

$$u_d \equiv \frac{u_{\max} - u^*}{u^* - u_{\min}} \quad (6.3)$$

$$v_d \equiv \frac{v_{\max} - v^*}{v^* - v_{\min}} \quad (6.4)$$

On this measure, the series is deep (tall) if the measure is less (greater) than unity.

In addition to the turning-point measures of business-cycle asymmetry, defined above, we calculate Sichel's (1993) measures of steepness and deepness, which are based upon the skewness of a series' first differences and levels, respectively. An advantage of Sichel's measures, when applied to real-world data, is that they avoid the need to accurately identify business-cycle turning points. His measures have gained some popularity in the literature on business-cycle asymmetry and it is interesting to compare them with more traditional measures based on turning points. Thus to measure the deepness of the stationary cyclical component, x_t , of a series he suggests computing

$$D_{(x)} = \frac{\sum_i (x_i - \bar{x})^3}{N\sigma(x)^3}, \tag{6.5}$$

where \bar{x} and $\sigma(x)$ are respectively the mean and standard deviation of x_t , and N is the sample size. If the series is deep (tall) it will have negative (positive) skewness, that is, $D(x) < (>)0$.

The measure of steepness is computed using the cyclical component's first difference, Δx_t :

$$ST(\Delta x_t) = \frac{\sum_i (\Delta x_t - \overline{\Delta x})^3}{N\sigma(\Delta x)^3}, \tag{6.6}$$

where $\overline{\Delta x}$ and $\sigma(\Delta x)$ are the mean and standard deviation of Δx . For a steep series (one whose contractions are steep) its first differences will have negative skewness, that is, $ST(\Delta x) < 0$; for a series which exhibits expansionary steepness, the first differences' skewness will be positive, $ST(\Delta x) > 0$.

Table 1: Guide to measures of solution-cycle characteristics

| | | |
|-----------------------------------|-------------------------|---|
| <i>Period and quarter-periods</i> | | |
| T | Total cycle period | |
| T_{iv} | } Quarter-cycle periods | } (u_{max}, v^*) to (u^*, v_{min}) (u^*, v_{min}) to (u_{min}, v^*) (u_{min}, v) to (u^*, v_{max}) (u^*, v_{max}) to (u_{max}, v^*) |
| T_{iii} | | |
| T_{ii} | | |
| T_i | | |
| <i>Turning-point measures</i> | | |
| A_u | Wage-share amplitude | } Peak-to-trough distance |
| A_v | Empl.-prop. amplitude | |
| u_d | Wage-share deepness | } Ratio of peak height to trough depth |
| v_d | Empl.-prop. deepness | |
| u_{st} | Wage-share steepness | } Ratio of expansion grad. to contraction grad. |
| v_{st} | Empl.-prop. steepness | |
| <i>Sichel's measures</i> | | |
| $D(u)$ | Wage-share deepness | } Skewness of series' levels |
| $D(v)$ | Empl.-prop. deepness | |
| $ST(\Delta u)$ | Wage-share steepness | } Skewness of series' first differences |
| $ST(\Delta v)$ | Empl.-prop. steepness | |

We are thus able to calculate wage-share and employment-proportion deepness, $D(u)$ and $D(v)$, by substituting into equation (6.5) the $N = T/\Delta t$ values of u and v recorded over the cycle. Then by substituting first differences of these values into equation (6.6), we can calculate Sichel's measures of wage-share and employment-proportion steepness, $ST(\Delta u)$ and $ST(\Delta v)$. Variable names for solution-cycle characteristics are summarised in Table 1.

The reduced form of our model (4.1)–(4.2) has ten independent structural parameters: a_1 , b_1 , a_2 and b_2 (derived from α , β , γ , ρ and σ , all of which have a counterpart in Goodwin's model); and the six tuning parameters, κ_1 , κ_2 , μ_1 , η_1 , μ_2 and η_2 . The model's solution trajectories will depend not only upon the values of these ten parameters but also upon the initial conditions, u_0 and v_0 . Given that there are also several measures of the solution characteristics, an exhaustive investigation of the model is precluded by the enormity of possible parametric permutations. Accordingly, we simplify the task somewhat by holding constant α' , β' , γ' , ρ' and σ' , and hence the four reduced-form parameters a_1 , b_1 , a_2 and b_2 , which derive from them. Thus the stationary point of economic interest, $(u^*, v^*) = (a_2/b_2, a_1/b_1) = (1 - (\alpha' + \beta')\sigma', (\alpha' + \gamma')/\rho')$, also remains invariant. These values are reported in Table 2. As we noted in section 5, above, the primed parameters are only basis values for the true parameters, $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$, $\hat{\rho}$ and $\hat{\sigma}$, which also depend upon the values of the tuning parameters and state variables. As we shall see in section 7, the values chosen result in economically realistic values of the true parameters in some of the simulations; they also imply economically realistic equilibrium values of the wage share and employment proportion—respectively, 60 per cent and 94 per cent.

Table 2: Structural parameter values

| | | |
|-------------------|---------------|-----------------|
| $\alpha' = 0.10$ | $a_1 = 15.10$ | $u^* = 0.60$ |
| $\beta' = 0.10$ | $b_1 = 16.00$ | $v^* = 0.94375$ |
| $\gamma' = 15.00$ | $a_2 = 0.30$ | |
| $\rho' = 16.00$ | | |
| $\sigma' = 2.00$ | | |

We perform simulations by varying both the tuning-parameter values and the initial conditions. For each simulation we report: total cycle period, T ; phase-variable amplitudes, A_u and A_v ; and the turning-point and skewness measures of asymmetry, which we described above and have summarised in Table 1. In addition, we calculate and report the approximation to the cycle's period, T_{approx} , based on the linearised system and given by equation (4.15), which is valid only close to (u^*, v^*) .

7 SIMULATION RESULTS

We begin by simulating Goodwin's original specification of the model, given by the system (2.9)–(2.10) (or, alternatively, by (3.9)–(3.10), with $\kappa_1 = \kappa_2 = 1$ and $\mu_1 = \eta_1 = \mu_2 = \eta_2 = 0$), choosing twelve different pairs of initial conditions. In all simulations, we use a step-size of $\Delta t = 0.1$, which gives a relative error in both $u(t)$ and $v(t)$ of 0.01 per cent or approximately 9 hours per 10-year cycle).¹³ Results of these simulations are summarised in Table 3, and three solution trajectories (and their corresponding time series) are plotted in Figure 2.

Table 3: Characteristics of Goodwin's model

| (u_0, v_0) | T | A_u % pts | u_{max} % | A_v % pts | v_{max} % | u_d | $D(u)$ | u_{st} | $ST(\Delta u)$ | v_d | $D(v)$ | v_{st} | $ST(\Delta v)$ |
|--------------|------|----------------|----------------|----------------|----------------|-------|--------|----------|----------------|-------|--------|----------|----------------|
| (0.53,0.87) | 2.99 | 70.55 | 102.03 | 15.50 | 102.34 | 1.47 | 0.41 | 1.06 | 0.06 | 1.03 | 0.06 | 0.78 | -0.41 |
| (0.54,0.88) | 2.98 | 60.57 | 95.30 | 13.34 | 101.20 | 1.40 | 0.35 | 1.05 | 0.05 | 1.03 | 0.05 | 0.81 | -0.35 |
| (0.55,0.89) | 2.96 | 50.75 | 88.91 | 11.20 | 100.08 | 1.32 | 0.30 | 1.04 | 0.04 | 1.03 | 0.04 | 0.84 | -0.30 |
| (0.56,0.90) | 2.97 | 41.05 | 82.85 | 9.07 | 98.98 | 1.26 | 0.24 | 1.03 | 0.04 | 1.02 | 0.04 | 0.87 | -0.24 |
| (0.57,0.91) | 2.96 | 31.47 | 77.11 | 6.97 | 97.90 | 1.19 | 0.18 | 1.02 | 0.03 | 1.01 | 0.03 | 0.90 | -0.18 |
| (0.58,0.92) | 2.96 | 22.00 | 71.67 | 4.87 | 96.83 | 1.13 | 0.13 | 1.02 | 0.02 | 1.01 | 0.02 | 0.92 | -0.13 |
| (0.59,0.93) | 2.95 | 12.63 | 66.54 | 2.80 | 95.78 | 1.07 | 0.08 | 1.01 | 0.01 | 1.01 | 0.01 | 0.95 | -0.08 |
| (0.60,0.94) | 2.96 | 3.39 | 61.71 | 0.75 | 94.75 | 1.02 | 0.02 | 1.00 | 0.00 | 1.00 | 0.00 | 0.99 | -0.02 |
| (0.60,0.96) | 2.96 | 14.58 | 67.59 | 3.23 | 96.00 | 1.08 | 0.09 | 1.01 | 0.01 | 1.01 | 0.01 | 0.95 | -0.09 |
| (0.61,0.97) | 2.96 | 23.57 | 72.56 | 5.22 | 97.01 | 1.14 | 0.14 | 1.02 | 0.02 | 1.01 | 0.02 | 0.92 | -0.14 |
| (0.62,0.98) | 2.96 | 32.60 | 77.77 | 7.21 | 98.03 | 1.20 | 0.19 | 1.03 | 0.03 | 1.01 | 0.03 | 0.90 | -0.19 |
| (0.63,0.99) | 2.97 | 41.62 | 83.20 | 9.20 | 99.05 | 1.26 | 0.24 | 1.03 | 0.04 | 1.02 | 0.04 | 0.87 | -0.24 |

We note from Figure 2 the existence of cycles which stray outside U when initial values (u_0, v_0) are sufficiently far from (u^*, v^*) . Note in particular that this economically unrealistic outcome is not merely the result of starting the cycle close to the boundary of U , since it can clearly arise when $u_0 < u$ and $v_0 < v$.

Using (2.11) we calculate the period of the linear approximation to the cycle as 2.95 (years). This figure should be compared to the values obtained numerically, from the full nonlinear cycle, in column 2 of Table 3, which summarises results of integrations commencing from different initial conditions (u_0, v_0) , all other parameters remaining fixed. We observe that the relative error in the linear approximation is always less than 1 per cent for realistic cycles, i.e. those contained within U . In Table 3 we also note that the turning-point measure of employment-proportion deepness, v_d of (6.4), remains close to unity throughout; Goodwin's original model therefore predicts that the employment-proportion oscillates with near-equal amplitude about its equilibrium value v^* . By contrast, u_d (of (6.3)) shows a marked increase above unity, indicating that wage share is a *tall* series. That u_{st} (of (6.1)) varies little from unity indicates that wage share contracts and expands

at approximately equal rates. By contrast, v_{st} (of (6.2)) falls noticeably below unity, particularly for the larger amplitudes A_v , so that employment proportion tends to expand more slowly than it contracts. That is, employment proportion is (down-)steep, whilst wage share is not steep. These observations on the series' steepness and deepness are corroborated via the skewness measures $D(u)$, $D(v)$, $ST(\Delta u)$ and $ST(\Delta v)$ presented in Table 3.

Finally, we should note that since we have set $\kappa_1 = \kappa_2 = 1$ and $\mu_1 = \eta_1 = \mu_2 = \eta_2 = 0$, the 'true' parameter values for productivity and labour-force growth, capital-output ratio and Phillips-curve parameters are equal to their basis values (the primed parameters). Clearly, in this case, these values (shown in Table 2) are not at all realistic. However, simulation results for Goodwin's model are not substantially altered when more economically reasonable values are chosen. For example, let us take $\hat{\alpha} = a' = 0.03$, $\hat{\beta} = \beta' = 0.01$, $\hat{\sigma} = \sigma' = 3$, $\hat{\gamma} = \gamma' = 0.63$ and $\hat{\rho} = \rho' = 0.7$. Then, for all initial points (u_0, v_0) given in Table 3, solution trajectories stray outside U , the economically feasible region, with $u_{max} > 120\%$ and $v_{max} > 115$ per cent in all cases, that is, the problem of the state variables attaining values which exceed unity is actually more severe.¹⁴

We now consider the first extension, given by equations (3.5)–(3.6) (or, alternatively, by (3.9)–(3.10), with $\mu_1 = \eta_1 = \mu_2 = \eta_2 = 0$). We choose $\kappa_1 = \kappa_2 = 5$, since this gives solution cycles whose periods are of the same order of magnitude as real-life business cycles (approximately 4–8 years); the same time step and initial conditions are again used. Results of these simulations are summarised in Table 4, and three solution trajectories (and their corresponding time series) are plotted in Figure 3.

Table 4: Simulation results for the first extension
($\mu_1 = \mu_2 = 0$, $\eta_1 = \eta_2 = 1$), with $\kappa_1 = \kappa_2 = 5$.

| (u_0, v_0) | T | A_u % pts | u_{max} % | A_v % pts | v_{max} % | u_d | $D(u)$ | u_{st} | $ST(\Delta u)$ | v_d | $D(v)$ | v_{st} | $ST(\Delta v)$ |
|--------------|------|----------------|----------------|----------------|----------------|-------|--------|----------|----------------|-------|--------|----------|----------------|
| (0.53,0.87) | 5.16 | 91.36 | 98.27 | 11.39 | 98.35 | 0.72 | -0.32 | 0.54 | -0.68 | 0.69 | -1.01 | 1.26 | 0.37 |
| (0.54,0.88) | 4.91 | 87.15 | 97.06 | 10.09 | 98.05 | 0.74 | -0.30 | 0.57 | -0.60 | 0.71 | -0.84 | 1.23 | 0.33 |
| (0.55,0.89) | 4.66 | 81.22 | 95.17 | 8.74 | 97.71 | 0.76 | -0.27 | 0.62 | -0.52 | 0.75 | -0.68 | 1.20 | 0.29 |
| (0.56,0.90) | 2.22 | 73.06 | 92.31 | 7.33 | 97.30 | 0.79 | -0.24 | 0.67 | -0.43 | 0.78 | -0.53 | 1.18 | 0.25 |
| (0.57,0.91) | 4.24 | 62.14 | 88.14 | 5.84 | 96.83 | 0.83 | -0.20 | 0.72 | -0.35 | 0.81 | -0.39 | 1.14 | 0.21 |
| (0.58,0.92) | 4.10 | 47.96 | 82.30 | 4.26 | 96.26 | 0.87 | -0.14 | 0.79 | -0.25 | 0.86 | -0.27 | 1.09 | 0.15 |
| (0.59,0.93) | 4.00 | 30.18 | 74.45 | 2.57 | 95.57 | 0.92 | -0.09 | 0.87 | -0.15 | 0.92 | -0.15 | 1.06 | 0.09 |
| (0.60,0.94) | 3.97 | 8.82 | 64.36 | 0.73 | 94.73 | 0.98 | -0.03 | 0.96 | -0.04 | 0.98 | -0.04 | 1.02 | 0.03 |
| (0.60,0.96) | 4.04 | 41.39 | 79.46 | 3.61 | 96.00 | 0.89 | -0.12 | 0.82 | -0.21 | 0.89 | -0.22 | 1.08 | 0.12 |
| (0.61,0.97) | 4.30 | 66.26 | 89.75 | 6.36 | 97.00 | 0.82 | -0.21 | 0.70 | -0.37 | 0.81 | -0.44 | 1.15 | 0.22 |
| (0.62,0.98) | 4.86 | 86.31 | 96.80 | 9.87 | 98.00 | 0.74 | -0.29 | 0.58 | -0.58 | 0.72 | -0.81 | 1.23 | 0.33 |
| (0.63,0.99) | 6.11 | 97.65 | 99.71 | 15.12 | 99.00 | 0.69 | -0.37 | 0.44 | -0.89 | 0.62 | -1.62 | 1.33 | 0.47 |

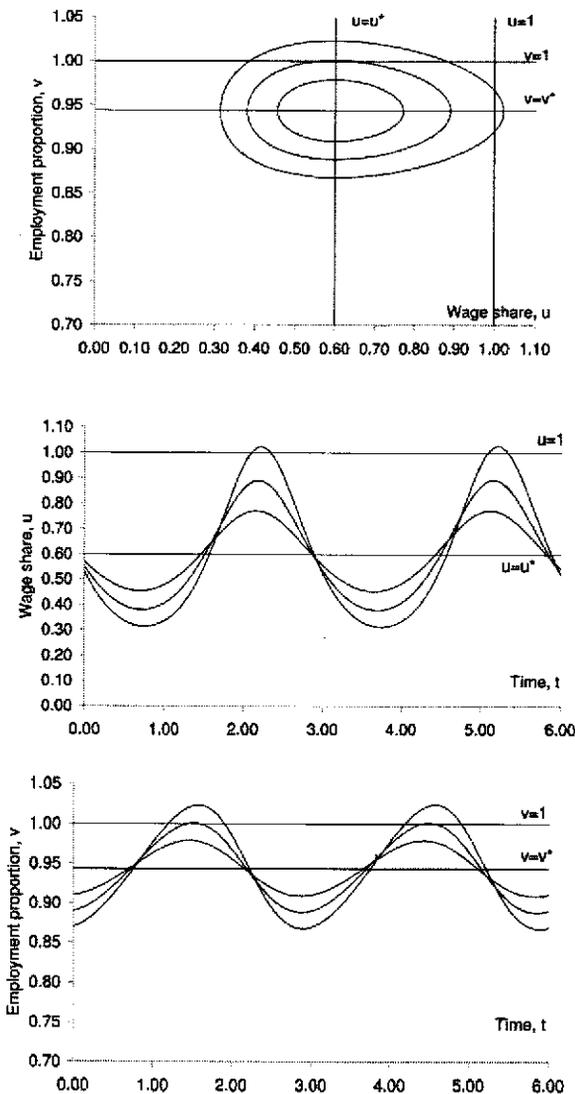


Figure 2: Goodwin's model: schematic solution trajectories for alternative initial conditions.

Top, middle and bottom sub-figures respectively show $u-v$ phase plane, u time series and v time series. Equilibrium values are respectively denoted by u^* and v^* . Note how both u and v exceed unity.

It is immediately evident from Figure 3 and Table 4 that all trajectories now remain within the economically feasible region, U , as expected. It is also clear that the time series of $u(t)$ and $v(t)$ have changed markedly.

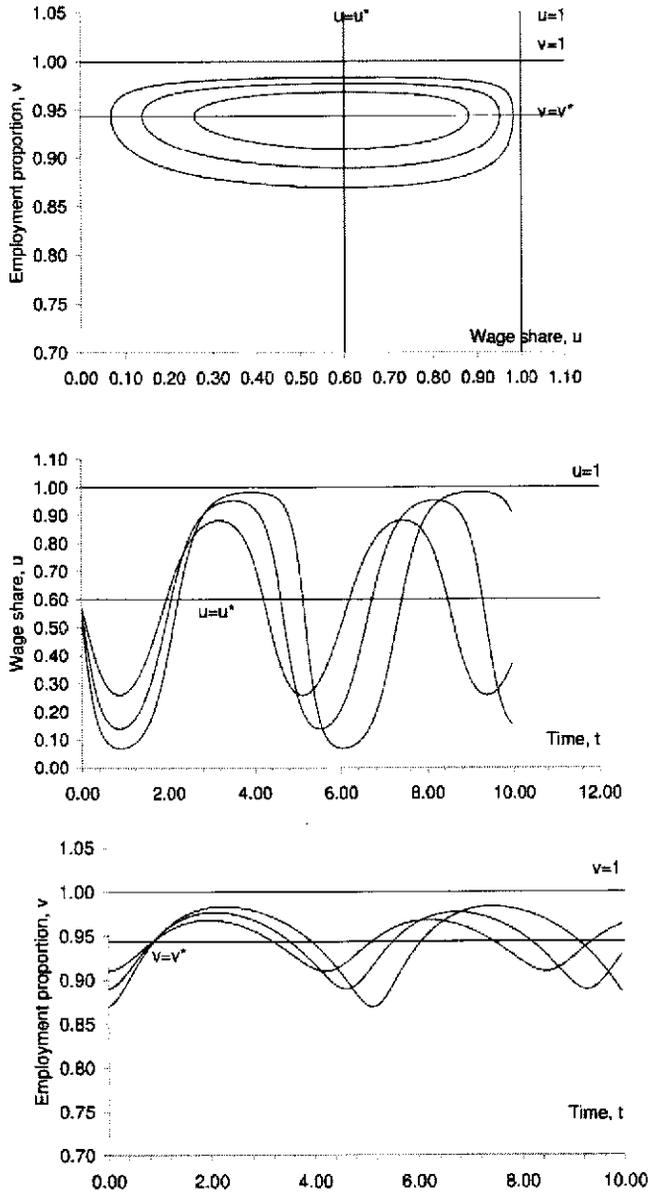


Figure 3: First extension, with $\kappa_1 = \kappa_2 = 5$: solution trajectories for alternative initial conditions.

Top, middle and bottom sub-figures respectively show $u-v$ phase plane, u time series and v time series. Equilibrium values are respectively denoted by u^* and v^* .

Using (4.15) we now calculate the period of the linear approximation to the cycle as 3.94 (years). This figure should be compared to the values obtained numerically, from the full nonlinear cycle, in column 2 of Table 4, which again summarises results of integrations commencing from the initial conditions (u_0, v_0) . What is noteworthy in these simulations is that the increased nonlinearity of the system (relative to the Goodwin specification) erodes the accuracy of the approximate cycle period far more rapidly as the cycle's amplitudes, A_u and A_v , increase. Considering the series' asymmetry properties, we see that both wage share and employment proportion are *deep* series—the turning point measures, u_d and v_d , are both less than unity, whilst the skewness measures, $D(u)$ and $D(v)$ are both negative. We can also see that the wage share series is (down-)steep— $u_{st} < 1$ and $ST(\Delta u) < 0$ —i.e., contractions are more rapid than expansions, whilst employment proportion, in a reversal of the situation in Goodwin's specification, is an up-steep series— $v_{st} > 1$ and $ST(\Delta v) > 0$ —i.e., expansions are more rapid than contractions. The general observation can be made that both u_{st} and v_{st} can now differ from unity and, similarly, $ST(\Delta u)$ and $ST(\Delta v)$ can differ from zero, by a greater amount than that possible in Goodwin's model; marked asymmetry is now evident, and this theme is developed further in the ensuing discussion.

In overcoming the unit-square problem via the inclusion of additional terms $(1 - u)$ in the expression for \dot{u}/u , and $(1 - v)$ in the expression for \dot{v}/v , we introduce an economic factor which merits more detailed consideration. At the stationary point (u^*, v^*) , we find that $\Phi \equiv (1 - u^*)/(1 - v^*) \gg 1$ as a result of the economic fact that v is realistically always much closer to unity than u . Thus, in the neighbourhood of the equilibrium point, the magnitude of wage-share growth is larger than employment-proportion growth, *ceteris paribus*, by a factor which is approximately Φ . This imbalance can be addressed by altering the parameters κ_1 and κ_2 or, more specifically, their ratio.

We use three initial conditions: $(u_0, v_0) = (0.64, 0.95)$, $(0.61, 0.97)$ and $(0.59, 0.93)$. We fix the product $\kappa_1 \kappa_2$ to be a constant, so that, via (4.15), the approximation to the period remains constant. Specifically, we choose $\kappa_1 \kappa_2 = 25$ — in keeping with the set of simulations just discussed—and $\kappa_1/\kappa_2 \in \{1, \frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \frac{1}{256}\}$. Accordingly, the approximation to the period is 3.94 years.

Results of this set of simulations are summarised in Table 5, and three solution trajectories (and their corresponding time series) are plotted in Figure 4 (in which $(u_0, v_0) = (0.64, 0.95)$).

The u - v phase plane in Figure 4 shows that the ratio A_u/A_v decreases with decreasing κ_1/κ_2 , even though column 2 of Table 5 reveals that the period T remains constant. The term κ_1 appears as a positive factor in the expression for wage-share growth, and κ_2 is a positive factor in the expression for employment-proportion growth. Thus, *ceteris paribus*, if κ_1/κ_2 increases, say,

the ratio $\frac{u}{v}$ will also increase: the cycle will therefore become stretched in the u -direction. If, on the other hand, κ_1/κ_2 falls, then $\frac{u}{v}$ will fall at all points on the cycle, which be stretched in the v -direction. In these simulations, the smallest value of κ_1/κ_2 gives the most economically realistic solution cycle, in the sense of the magnitudes of the wage-share and employment-proportion amplitudes; decreasing κ_1/κ_2 further does not alter the portrait significantly.

Table 5: Characteristics of first extension for alternative ratios κ_1/κ_2 , with $\kappa_1\kappa_2 = 25$

| κ_1/κ_2 | T 'years' | A_u % pts | A_v % pts | u_d | $D(u)$ | u_{st} | $ST(\Delta u)$ | v_d | $D(v)$ | v_{st} | $ST(\Delta v)$ |
|-----------------------------|----------------|----------------|----------------|-------|--------|----------|----------------|-------|--------|----------|----------------|
| $(u_0, v_0) = (0.59, 0.93)$ | | | | | | | | | | | |
| 1 | 4.00 | 30.18 | 2.57 | 0.92 | -0.11 | 0.82 | -0.20 | 0.87 | -0.20 | 1.00 | 0.12 |
| 1/4 | 4.00 | 15.46 | 2.59 | 0.96 | -0.07 | 0.82 | -0.19 | 0.87 | -0.20 | 1.00 | 0.08 |
| 1/16 | 4.00 | 7.96 | 2.65 | 0.98 | -0.06 | 0.82 | -0.18 | 0.86 | -0.19 | 1.00 | 0.06 |
| 1/64 | 4.00 | 4.34 | 2.89 | 0.99 | -0.04 | 0.82 | -0.19 | 0.85 | -0.20 | 1.00 | 0.05 |
| 1/256 | 4.00 | 2.77 | 3.69 | 0.99 | -0.03 | 0.82 | -0.23 | 0.81 | -0.24 | 1.00 | 0.03 |
| $(u_0, v_0) = (0.61, 0.97)$ | | | | | | | | | | | |
| 1 | 4.40 | 66.26 | 6.36 | 0.82 | -0.21 | 0.83 | -0.42 | 0.70 | -0.35 | 1.20 | 0.17 |
| 1/4 | 4.00 | 36.81 | 6.37 | 0.90 | -0.08 | 0.82 | -0.41 | 0.70 | -0.38 | 1.00 | 0.07 |
| 1/16 | 4.00 | 18.99 | 6.40 | 0.95 | -0.04 | 0.82 | -0.39 | 0.70 | -0.38 | 1.00 | 0.02 |
| 1/64 | 4.00 | 9.72 | 6.50 | 0.97 | -0.02 | 0.82 | -0.38 | 0.70 | -0.38 | 1.00 | -0.01 |
| 1/256 | 4.40 | 5.17 | 6.90 | 0.99 | 0.00 | 0.82 | -0.40 | 0.68 | -0.39 | 1.00 | -0.03 |
| $(u_0, v_0) = (0.64, 0.95)$ | | | | | | | | | | | |
| 1 | 3.80 | 17.44 | 1.46 | 0.95 | -0.10 | 0.90 | -0.06 | 0.92 | -0.08 | 1.11 | 0.09 |
| 1/4 | 4.00 | 11.21 | 1.87 | 0.97 | -0.07 | 0.82 | -0.14 | 0.90 | -0.16 | 1.00 | 0.09 |
| 1/16 | 4.00 | 8.98 | 2.99 | 0.98 | -0.05 | 0.82 | -0.20 | 0.85 | -0.21 | 1.00 | 0.06 |
| 1/64 | 4.00 | 8.32 | 5.56 | 0.98 | -0.02 | 0.82 | -0.33 | 0.73 | -0.33 | 1.00 | 0.01 |
| 1/256 | 3.80 | 8.15 | 10.96 | 0.98 | -0.03 | 0.69 | -0.63 | 0.55 | -0.64 | 1.00 | -0.04 |

An effect of altering the ratio κ_1/κ_2 is as follows. Suppose that T_1 and T_2 are trajectories passing through (u_1, v_1) and (u_2, v_2) respectively, where (u_1, v_1) , (u_2, v_2) and (u^*, v^*) are distinct. For certain values of κ_1/κ_2 , T_1 will be contained within T_2 — since the governing equations are autonomous, T_1 and T_2 can never intersect — whereas for other such values, T_2 will be contained within T_1 . This feature accounts for the wide variability in both A_u and A_v in Table 5. The remaining information in Table 5 reveals that changing the ratio κ_1/κ_2 has negligible effect on the asymmetry characteristics of the cycles. In order to model such features more realistically, we now move to our second extension and address variation of the parameters μ_1, μ_2, η_1 and η_2 .

We conduct our simulations using parameter values chosen from $\mu_1, \mu_2 \in \{0, 1, 2, 3\}$ and $\eta_1, \eta_2 \in \{1, 2, 3\}$, giving a set of 144 possible combinations.

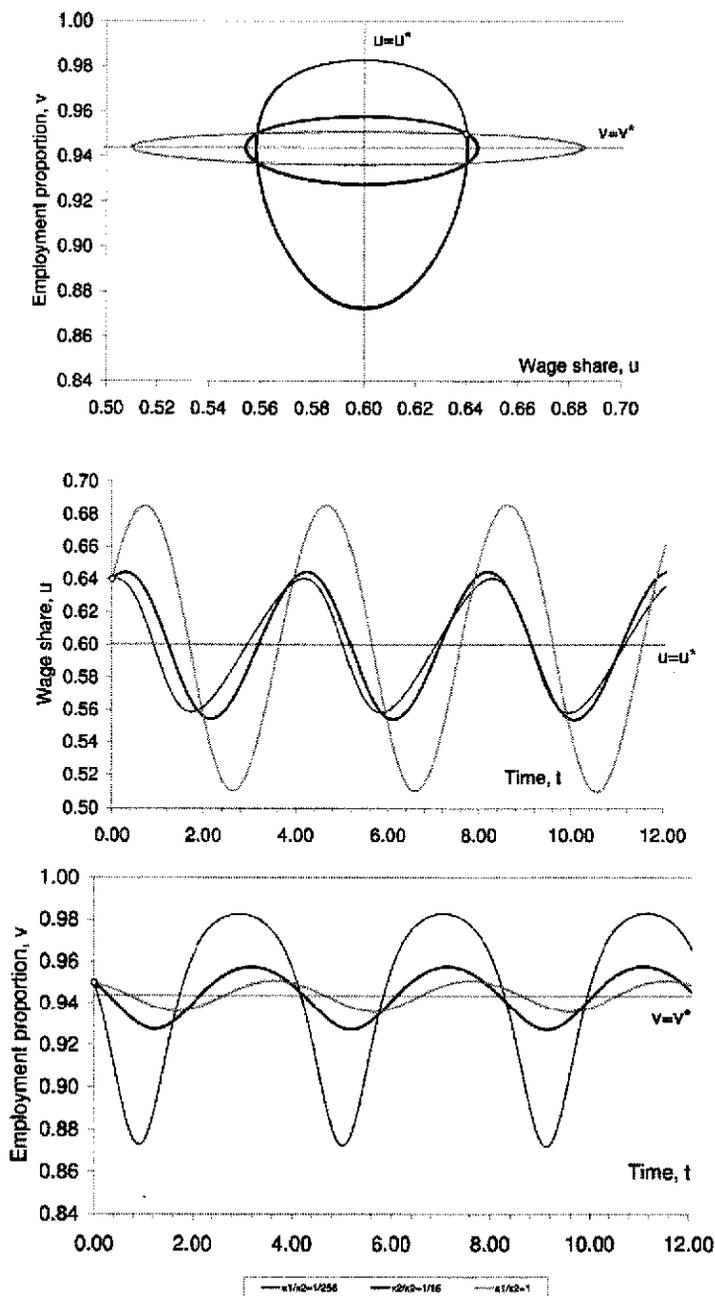


Figure 4: First extension: solution trajectories for varying parameter ratios: $1/2= 1/256, 1/16$ and $1; 12 = 25$ throughout.

Top, middle and bottom sub-figures respectively show $u-v$ phase plane, u time series and v time series.

There is no theoretical reason why the parameters μ_1 , μ_2 , η_1 and η_2 should take integer values. It should be noted, however, that the ratios $\mu_1/(\mu_1 + \eta_1)$ and $\mu_2/(\mu_2 + \eta_2)$ determine the respective values of u and v at which $f_1(u)$ and $f_2(v)$ are maximised and which separate neo-Keynesian regime, say, from social conflict regime — see the discussion in section 5 above.

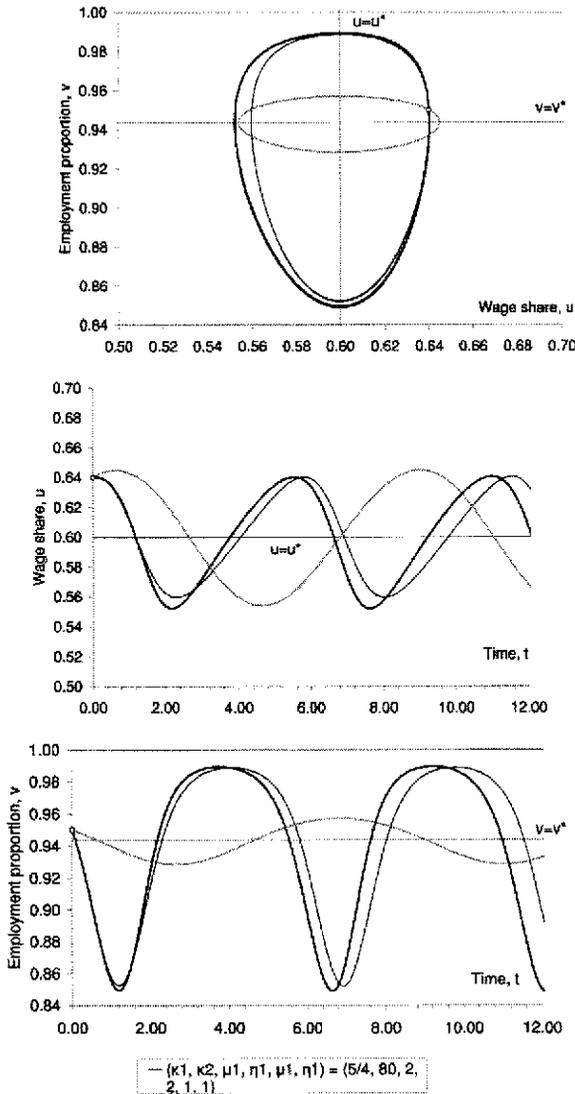


Figure 5: Second extension: solution trajectories for alternative tuning parameter values.

Top, middle and bottom sub-figures respectively show u - v phase plane, u time series and v time series.

Note also that the model just considered (the first extension) merely employs the parameters $\mu_1 = \mu_2 = 0$ and $\eta_1 = \eta_2 = 1$. For each of the above-mentioned 144 combinations, nine simulations were undertaken using (κ_1, κ_2) pairs chosen so as to give $\kappa_1/\kappa_2 \in \{\frac{1}{4}, \frac{1}{16}, \frac{1}{64}\}$, and initial conditions chosen from $(u_0, v_0) \in \{(0.64, 0.95), (0.59, 0.93), (0.61, 0.97)\}$. The choices for the ratio κ_1/κ_2 were guided by the discussion presented above. Exact values of κ_1 and κ_2 were chosen so as to give economically-plausible business-cycle periods.¹⁵

Given the considerable number of parameter combinations, and also the fact that we are interested in several aspects of the resulting model's behaviour, there is no easy way to summarise our simulation results. We present in Table 6 results of a subset of these simulations which are representative of the model's varied behaviour; full results for all nine simulation sets —including scatter plots, time series etc. — can be found in Harvie *et al.* (2001).

Table 6: Characteristics of second extension

| κ_1 | κ_2 | μ_1 | η_1 | μ_2 | η_2 | T | T_{approx} | A_u | A_v | u_D | $D(u)$ | u_{ST} | $ST(\Delta u)$ | v_D | $D(v)$ | v_{ST} | $ST(\Delta v)$ |
|-----------------------------|------------|---------|----------|---------|----------|--------|---------------------|-------|-------|-------|--------|----------|----------------|-------|--------|----------|----------------|
| | | | | | | years' | years' | % | % | pts | pts | | | | | | |
| $(u_0, v_0) = (0.59, 0.93)$ | | | | | | | | | | | | | | | | | |
| 20 | 80 | 3 | 1 | 0 | 2 | 4.55 | 4.46 | 28.21 | 2.43 | 1.44 | 0.39 | 0.84 | -0.31 | 0.76 | -0.30 | 0.78 | -0.40 |
| 5 | 320 | 3 | 1 | 0 | 2 | 4.50 | 4.46 | 7.32 | 2.52 | 1.11 | 0.11 | 0.84 | -0.31 | 0.75 | -0.31 | 0.94 | -0.11 |
| 5/4 | 80 | 2 | 2 | 1 | 1 | 5.40 | 5.34 | 2.51 | 4.27 | 1.00 | 0.00 | 0.86 | -0.24 | 0.80 | -0.24 | 1.00 | 0.00 |
| 5/4 | 80 | 0 | 3 | 1 | 1 | 5.10 | 5.06 | 2.53 | 4.08 | 0.95 | -0.05 | 0.89 | -0.21 | 0.81 | -0.21 | 1.04 | 0.05 |
| 5/8 | 10 | 0 | 1 | 2 | 1 | 8.40 | 8.34 | 16.51 | 2.66 | 0.98 | -0.05 | 0.91 | -0.11 | 0.87 | -0.12 | 1.10 | 0.03 |
| $(u_0, v_0) = (0.61, 0.97)$ | | | | | | | | | | | | | | | | | |
| 20 | 80 | 3 | 1 | 0 | 2 | 5.58 | 4.46 | 65.68 | 9.12 | 1.53 | 0.53 | 0.59 | -1.71 | 0.40 | -1.01 | 0.65 | -0.80 |
| 5 | 320 | 3 | 1 | 0 | 2 | 4.83 | 4.46 | 23.19 | 9.16 | 1.36 | 0.32 | 0.56 | -1.04 | 0.40 | -1.01 | 0.82 | -0.36 |
| 5/4 | 80 | 2 | 2 | 1 | 1 | 5.80 | 5.34 | 4.19 | 7.12 | 1.00 | -0.02 | 0.74 | -0.41 | 0.69 | -0.42 | 1.00 | 0.02 |
| 5/4 | 80 | 0 | 3 | 1 | 1 | 5.40 | 5.06 | 4.40 | 7.07 | 0.92 | -0.07 | 0.79 | -0.41 | 0.69 | -0.41 | 1.00 | 0.07 |
| 5/8 | 10 | 0 | 1 | 2 | 1 | 8.00 | 8.34 | 19.72 | 6.21 | 0.95 | -0.04 | 0.82 | -0.33 | 0.74 | -0.33 | 1.00 | 0.05 |
| $(u_0, v_0) = (0.64, 0.95)$ | | | | | | | | | | | | | | | | | |
| 20 | 80 | 3 | 1 | 0 | 2 | 4.48 | 4.46 | 17.66 | 1.50 | 1.27 | 0.24 | 0.90 | -0.18 | 0.84 | -0.18 | 0.87 | -0.26 |
| 5 | 320 | 3 | 1 | 0 | 2 | 4.48 | 4.46 | 8.58 | 2.97 | 1.13 | 0.13 | 0.81 | -0.36 | 0.72 | -0.36 | 0.93 | -0.13 |
| 5/4 | 80 | 2 | 2 | 1 | 1 | 5.80 | 5.34 | 8.04 | 13.61 | 1.00 | 0.01 | 0.66 | -0.74 | 0.49 | -0.74 | 1.00 | -0.02 |
| 5/4 | 80 | 0 | 3 | 1 | 1 | 5.40 | 5.06 | 8.77 | 13.97 | 0.84 | -0.19 | 0.64 | -0.74 | 0.48 | -0.74 | 1.16 | 0.20 |
| 5/8 | 10 | 0 | 1 | 2 | 1 | 8.40 | 8.34 | 9.07 | 2.85 | 0.96 | -0.01 | 0.91 | -0.10 | 0.87 | -0.09 | 1.10 | -0.01 |

Three solution trajectories (and their corresponding time series) are plotted in Figure 5. Table 6 shows that T_{approx} constitutes a good approximation to the simulated period T over a wide range of parameters, whilst we can see that T increases with increasing values of the parameters μ_1, η_1, μ_2 and η_2 , as predicted by (4.15). However, as we would also expect, the accuracy of T_{approx} declines as the cycle's amplitude increases. The relationship between

the model's parameters and the resulting cycle amplitudes, A_u and A_v , is far less clear: in fact, consideration of the full set of results reveals this relationship to be nonlinear, in the sense that the effect—its sign as well as its magnitude—on A_u and A_v of increasing, say, one parameter depends upon the values of the others (Harvie *et al.*, 2001).

Considering now the cycles' asymmetry properties, for the first time, we observe that the employment-proportion series can be either down-steep or up-steep (or neither), depending upon the parameter values. The wage share series, on the other hand, is always down-steep. In terms of business-cycle deepness, the wage share can be either deep or tall, while the employment proportion is always deep. In fact, we tend to observe wage share deepness (tallness) when the employment proportion is up-steep (down-steep). As for the previous model specifications, these summaries can be gleaned from either the turning-point or the skewness measures of asymmetry (Harvie *et al.*, 2001). Finally, we can note, from Figure 5, that employment-proportion peaks tend to be more rounded than troughs, reflecting the expectation of short-lived depressions and prolonged prosperity, in keeping with Keynes' (1936) observation concerning business-cycle turning points.

Finally, we can consider the implied 'true' values of productivity, labour-force growth, etc., which can be computed from equations (5.6)–(5.10) in section 5, above.¹⁶ It is noteworthy that, whilst for some simulations these variables take on widely unrealistic values, for others their values are economically plausible. For example, for $(\kappa_1, \kappa_2) = (5/8, 10)$ and $(\mu_1, \eta_1, \mu_2, \eta_2) = (0, 1, 2, 1)$, productivity growth varies between 2 per cent and 3 per cent over the cycle, the combined effect of productivity and labour-force growth is 3–12 per cent, the Phillips-curve slope is takes values between 3.5 and 4.5, whilst capital-output ratio varies in the range 3–5.

8 CONCLUDING COMMENTS

In this paper we have developed two extensions to Goodwin's (1967; 1972) growth-cycle model which ensure that solution trajectories remain in the economically feasible region. Besides resolving this unrealistic characteristic of Goodwin's original specification, our new model has a richer economic interpretation and allows the generation of asymmetric solution cycles and trajectories. We have used numerical methods to study in detail the characteristics of Goodwin's model and our extensions, paying particular attention to two forms of symmetry/asymmetry — businesscycle steepness and deepness — displayed by the models' state variables, u and v . For reasons of space, we have not been able to explore every aspect of the model's behaviour. For example, we did not investigate the effect of altering the underlying base values of productivity and labour-force growth, capital-output ratio and the Phillips-curve parameters, which give the equilibrium state (u^*, v^*) . Whilst we have concentrated on exploring the extent to which the wage share and employment proportion were asymmetric, we have not considered the symmetry or asym-

metry of other important economic variables, such as total employment, total output or or the rate of profit on capital, which can be easily derived from the model.

It should also be noted that our model shares some of the other weaknesses of Goodwin's, in particular, his neglect of questions of effective or aggregate demand. Explicitly modelling such effects is not possible in a two-dimensional model which retains wage share and employment proportion as state variables and we have chosen to remain closer to Goodwin in this respect. Nevertheless, as we attempted to show in section 5 our model does capture some effective demand effects, albeit indirectly. It is clear from our simulations that our new model can generate a rich variety of economic outcomes. Many of these are plausible which suggests further work calibrating this model would be worthwhile. The does have some restrictions, such as the close relationships between wage-share deepness and employment-proportion steepness, on the one hand, and wage-share steepness and employment-proportion deepness, on the other. The fact remains, however, that there is still little empirical evidence on business-cycle asymmetry, still less a concensus of opinion. Moreover, although researchers have considered the possible asymmetry of employment and unemployment, they have not investigated the asymmetry properties of the employment proportion, a key variable in both Goodwin's model and our extensions. Nor is there any empirical evidence on the symmetry or asymmetry on the Goodwin model's other state variable, the wage share. We turn our attention to these empirical questions in future research.

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ENDNOTES

1. School of Management, University of Leicester, Leicester LE1 7RH, England. dh98@leicester.ac.uk (Harvie — corresponding author); Department of Applied Mathematics, University of Leeds, Leeds LS2 9JT, England. mark@maths.leeds.ac.uk (Kelmanson) and d.g.knapp@leeds.ac.uk (Knapp). We are grateful to Peter Skott, Kevin Reilly, Malcolm Sawyer and an anonymous referee for useful comments on earlier drafts of this paper. Remaining errors are our own, of course. DH acknowledges the financial support of ESRC studentship number R00429724487.
2. Figures arising through authors' calculations, based on NBER business-cycle reference dates reported in Stock and Watson (1998).
3. Sichel uses post-war quarterly US unemployment, real GNP and industrial production data; Speight uses monthly industrial production indices for 16 OECD countries; Speight and McMillan use 27 UK macroeconomic time series.
4. A difference in shape of a cycle's peaks and troughs has been termed business-cycle sharpness by McQueen and Thorley (1993), who investigate the phenomenon empirically. It corresponds to asymmetries in the way an economy moves from contraction to expansion, on the one hand, and from expansion to contraction, on the other.

5. These issues are discussed and addressed by Skott (1989a,b), who integrates Goodwin's model with Kaldor's (1940) 'model of the trade cycle'. Velupillai (1983) also allows for a variable capital-output ratio and a more realistic investment function in his 'neo-Cambridge' model, whilst another dynamic business-cycle model combining a Keynesian-Kaleckian demand-driven economy with a Marxian/Goodwinian class-conflict driven one is proposed by Dibeh (1995).
6. May (1974) discusses a number of ways of making the simplest Lotka-Volterra predator-prey model more realistic.
7. This natural limit (of unity) to employment is analogous to the prey carrying capacity in biological predator-prey models. We can associate its effect with the $(1 - v)\eta_2$ term in $f_2(v)$.
8. We make this assumption since effective demand is not modelled explicitly.
9. It is possible to solve for $\hat{\beta}$: combining equations (5.6) and (5.10), we obtain:
$$\hat{\beta} = f_2(v)(\alpha' + \beta') - f_1(u)\alpha'$$
 This should simply be interpreted as an expression for the growth in potential output which is not explained by productivity improvements.
10. This is also the case for the other structural parameters, $\hat{\alpha}$ and $\hat{\beta}$, discussed above, and $\hat{\sigma}$, discussed below. In each case, two regimes are described, one in which the parameter's relationship with one of the state variables is positive, the other in which it is negative. The economy will tend to remain with a single regime during the course of one business cycle. To the extent that our model can capture the characteristics of two or more theoretical regimes, depending upon the location of the cycle in $u-v$ space, it shares some similarities with that of Bhaduri and Marglin (1990).
11. It should be stressed that we do not explicitly model effective demand and specify no investment function.
12. Note that, in general, a quarter period T_i will not be exactly one quarter of the full cycle period T .
13. For more details, see, for example, Gerald and Wheatley (1994).
14. These 'more realistic' parameter values also imply a cycle period of approximately 14 years, and an equilibrium wage share of $u = 0.88$, which exceeds the upper limit of wage-share fluctuations for OECD economies (Harvie, 2000).
15. *Ceteris paribus*, the cycle period varies inversely with $\sqrt{\kappa_1\kappa_2}$. It is thus straightforward to adjust these multiplicative terms, provided their ratio remains constant, to predictably alter T .
16. These values are not reported in Table 6.

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